Abstract Linear Algebra, Fall 2011 - Solutions to Problems I

1. *Determine in each case whether the given subset of $\mathbb{R}^3$ defines a subspace:*
   a. the set of all (x, y, z) such that $x + y + z = 0$;
   b. the set of all (x, y, z) such that $xyz = 0$;
   c. the set of all (x, y, z) such that $z = x + y$;
   d. the set of all (x, y, z) such that $xy = 1$.

   a and c define subspaces, b and d do not. For example, for b, the vectors $(0, 1, 1)$ and $(1, 0, 0)$ belong to the given set but their sum $(1, 1, 1)$ clearly does not. Similarly, for d, $(1, 1)$ and $(2, 1/2)$ are in the set but again their sum $(3, 3/2)$ is not. Alternatively, the zero vector $(0, 0)$ is not in d, so d cannot define a subspace. For a and c, check that the given sets are closed under addition and scalar multiplication. Further it is clear that they are non-empty and thus define subspaces.

2. *Let U and W be subspaces of a vector space V.*
   a. Show that $U \cap W$ and $U + W$ are subspaces of V.
   b. Is $U \cup W$ always a subspace of V?

   a. Let v_1 and v_2 be vectors in $U \cap W$. In particular, $v_1 \in U$ and $v_2 \in U$. Since U is a subspace, $v_1 + v_2 \in U$. Similarly, $v_1 \in W$ and $v_2 \in W$, and so $v_1 + v_2 \in W$. Thus $v_1 + v_2 \in U \cap W$. Now let α be a scalar. Then αv_1 is in U and in W, i.e., $\alpha v_1 \in U \cap W$. Thus $U \cap W$ is closed under addition and scalar multiplication. It is also non-empty (since the zero vector is in U and in W) and so is a subspace.

   Let v_1 and v_2 be vectors in $U + W$ so that $v_1 = u_1 + w_1$ and $v_2 = u_2 + w_2$ for vectors $u_1, u_2 \in U$ and vectors $w_1, w_2 \in W$. We have
   $$\begin{aligned} v_1 + v_2 &= u_1 + w_1 + u_2 + w_2 \\ &= (u_1 + u_2) + (w_1 + w_2). \end{aligned}$$
   Since $u_1 + u_2 \in U$ and $w_1 + w_2 \in W$, this shows that $v_1 + v_2 \in U + W$, i.e., $U + W$ is closed under addition. Similarly, if α is a scalar, then
   $$\begin{aligned} \alpha v_1 &= \alpha(u_1 + w_1) \\ &= \alpha u_1 + \alpha w_1. \end{aligned}$$
   Since $\alpha u_1 \in U$ and $\alpha w_1 \in W$, it follows that $\alpha v_1 \in U + W$, i.e., $U + W$ is closed under scalar multiplication. Finally, $U + W$ is clearly non-empty (for example, it contains U) and so $U + W$ is a subspace.

   b. It is not true in general that $U \cup W$ is a subspace of V. For instance, take $V = \mathbb{R}^2$ and let
   $$U = \{(x, 0) | x \in \mathbb{R}\}, \ W = \{(0, y) | y \in \mathbb{R}\}.$$
   So U is the x-axis and W is the y-axis in the plane. Then $(1, 0)$ is in U and $(0, 1)$ is in W but their sum $(1, 1)$ is not in U and not in W, i.e., is not in $U \cup W$. Thus $U \cup W$ is not closed under addition.

   In fact, given subspaces U and W of a vector space V, the set $U \cup W$ is a subspace of V if and only if $U \subset W$ or $W \subset U$. Can you prove this?

3. *Consider the real vector space of all real functions of a real variable t. Determine in each case whether the given functions are linearly independent:*

   a. $\cos t$, $\sin t$;

   b. $\cos^2 t$, $\sin^2 t$;

   c. $1, \cos 2t$, $\cos^2 t$;

   d. 1, t, e^t.

   a. Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha \cos t + \beta \sin t = 0$, for all $t \in \mathbb{R}$. Setting $t = 0$, we see $\alpha = 0$ (as $\cos 0 = 1$ and $\sin 0 = 0$). Thus the relation becomes $\beta \sin t = 0$, for all $t \in \mathbb{R}$. This clearly implies $\beta = 0$ (as $\sin t$ is not the zero function). Thus $\cos t, \sin t$ are linearly independent.
   b. $\cos^2 t, \sin^2 t$ are also linearly independent (exactly as in a).
   c. From the dim past (at least for me), we recall the trig. identity $\cos 2t = 2\cos^2 t - 1$. Equivalently,
   $$1 + \cos 2t - 2\cos^2 t = 0,$$
   for all real numbers t, and so the given functions are linearly dependent.
   d. Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha + \beta t + \gamma e^t = 0$, for all $t \in \mathbb{R}$. Differentiating this relation gives (i) $\beta + \gamma e^t = 0$ (for all t). Differentiating again gives (ii) $\gamma e^t = 0$ (for all t). Since e^t is not identically zero, (ii) implies that $\gamma = 0$ and then (i) implies $\beta = 0$. Substituting $\beta = 0$ and $\gamma = 0$ in the original relation, we see also that $\alpha = 0$, and so the given functions are linearly dependent.

4. *Let $\mathcal{P}_n$ denote the real vector space of all real polynomials of degree $\leq n$ (for n a non-negative integer) and write $\mathcal{P}_n^+$ for the subspace of even polynomials in $\mathcal{P}_n$, i.e., $\mathcal{P}_n^+ = \{p(t) \in \mathcal{P}_n : p(t) = p(-t)\}$. Find the dimensions of $\mathcal{P}_n$ and $\mathcal{P}_n^+$.*

   The space $\mathcal{P}_n$ is generated by $1, t, \ldots, t^n$. Further these elements are linearly indepedendent. Thus $1, t, \ldots, t^n$ is a basis of $\mathcal{P}_n$ and $\dim \mathcal{P}_n = n + 1$.

   Let $p(t) = \sum_{i=0}^{n} a_i t^i \in \mathcal{P}_n$. Then
   $$\begin{aligned} p(-t) &= \sum_{i=0}^{n} a_i (-t)^i \\ &= \sum_{i=0}^{n} (-1)^i a_i t^i. \end{aligned}$$
   Thus $p(t) = p(-t)$ if and only if $a_i = (-1)^i a_i$, for $i = 0, 1, \ldots, n$. Now $(-1)^i = \pm 1$ according as i is even or odd. It follows that $p(t) = p(-t)$ if and only if $a_i = -a_i$, or equivalently $a_i = 0$, for all odd indices i. Hence $\mathcal{P}_n^+$ is generated by the even powers of t in $\mathcal{P}_n$. Since the powers of t are linearly independent, we see that $\mathcal{P}_n^+$ has as a basis the elements $1, t^2, \ldots, t^{2k}$ where k is the largest integer such that $2k \leq n$. Thus $k = [n/2]$, the greatest integer $\leq n/2$, and so $\dim \mathcal{P}_n^+ = [n/2] + 1$.

5. *Let $v_1, \ldots, v_m$ be linearly independent vectors in a vector space V. Suppose w is a vector in V such that $v_1 + w, \ldots, v_m + w$ are linearly dependent. Show that w is a linear combination of $v_1, \ldots, v_m$.*

There exist scalars $\beta_1, \ldots, \beta_m$, not all zero, such that

$$\beta_1(v_1 + w) + \cdots + \beta_m(v_m + w) = 0.$$

Rearranging, this gives

$$\beta_1 v_1 + \cdots + \beta_m v_m = -(\beta_1 + \cdots + \beta_m)w. \qquad (*)$$

Note $\beta_1 + \cdots + \beta_m \neq 0$. [Indeed, if $\beta_1 + \cdots + \beta_m = 0$, then $(*)$ becomes

$$\beta_1 v_1 + \cdots + \beta_m v_m = 0.$$

Linear independence of $v_1, \ldots, v_m$ then forces $\beta_1 = 0, \ldots, \beta_m = 0$ which contradicts our hypothesis that $v_1 + w, \ldots, v_m + w$ are linearly dependent.] Since $\beta_1 + \cdots + \beta_m \neq 0$, we can rewrite $(*)$ as

$$-(\beta_1 + \cdots + \beta_m)^{-1}\beta_1 v_1 - \cdots - (\beta_1 + \cdots + \beta_m)^{-1}\beta_m v_m = w,$$

and so w is a linear combination of $v_1, \ldots, v_m$.

6. *Show that $\mathbb{R}$ has infinite dimension as a vector space over $\mathbb{Q}$. (One way to to do this is via the existence of real transcendental numbers.)*

Let α be a real transcendental number (for example, π or e). This means that α is not a root of any non-zero polynomial with rational coefficients. Explicitly, for any positive integer n and any rational numbers $a_0, a_1, \ldots, a_n$, with $a_i \neq 0$ for some i, we have

$$a_0 + a_1\alpha + \ldots + a_n\alpha^n \neq 0.$$

Equivalently, the $n + 1$ vectors $1, \alpha, \ldots, \alpha^n$ are linearly independent in $\mathbb{R}$ viewed as a vector space over $\mathbb{Q}$.
Now recall from class that in a vector space of dimension N any set of linearly independent vectors has at most N elements. We've just noted that $\mathbb{R}$ as a vector space over $\mathbb{Q}$ contains a set of linearly independent vectors of size $n + 1$, for *any* positive integer n. Hence $\mathbb{R}$ cannot have finite dimension as a vector space over $\mathbb{Q}$. That is, $\mathbb{R}$ has infinite dimension as a vector space over $\mathbb{Q}$.

Alternative Solution. Here's another approach using the concept of countability which you may have seen in Math 2513. The key point is that $\mathbb{R}$ is uncountable whereas $\mathbb{Q}^n$ is countable (for any positive integer n). Suppose $\mathbb{R}$ has finite dimension as a vector space over $\mathbb{Q}$ and let $r_1, \ldots, r_n$ be a basis for this vector space. Thus, given $r \in \mathbb{R}$, there exist unique rational numbers $q_1, \ldots, q_n$ such that $r = q_1 r_1 + \cdots + q_n r_n$. In other words, the map

$$(q_1, \ldots, q_n) \mapsto q_1 r_1 + \cdots + q_n r_n : \mathbb{Q}^n \to \mathbb{R}$$

is a bijection. But this is impossible: $\mathbb{Q}^n$ is countable and $\mathbb{R}$ is uncountable and so there cannot be a bijection between these sets. We conclude that $\mathbb{R}$ must have infinite dimension as a vector space over $\mathbb{Q}$.