Cycles and Permutations

A *transposition* is a 2-cycle.

Notice how any m-cycle can be written as a product (composition) of $m-1$ transpositions:

$$(i_1, i_2, \ldots, i_m) = (i_1, i_m)(i_1, i_{m-1}) \cdots (i_1, i_2).$$

Now this way of writing a cycle as a product of transpositions is just one way of doing so. There are many different ways.

Examples:

$$(2,5,3,6) = (2,6)(2,3)(2,5) = (5,2)(3,5)(6,3) = (1,7)(2,6)(2,3)(2,5)(1,7).$$

$$(1) = (1,3)(1,3) = (1,2)(1,2)(3,5)(3,5).$$

What we want to show is that, while a cycle (or any type of permutation) may be written in different ways as a product of transpositions, and even using different numbers of transpositions, the *parity* of the number of transpositions used stays the same. In other words, if a permutation can be written as the product of an odd number of transpositions, then it can only be written as the product of an odd number of transpositions. If a permutation can be written as the product of an even number of transpositions, then it can only be written as a product of an even number of transpositions.

How could we do this? How should we think about doing something like this?

What we need is a way to think of permutations as either "even" or "odd." In other words, we need a suitable **function** which takes permutations as input and has either 1 or -1, say, as output.

Start with n variables: $X_1, X_2, \dots, X_n$. Take the product

$$\prod_{i<j}(X_i - X_j)$$

and call this product P. When $n = 4$, this product is

$$P = (X_1 - X_2)(X_1 - X_3)(X_1 - X_4)(X_2 - X_3)(X_2 - X_4)(X_3 - X_4).$$

Generally speaking, there will be $n(n-1)/2$ factors in this product. For every pair of variables X_i and X_j, either $X_i - X_j$ or $X_j - X_i$ is a factor of P, but not both.

Since the subscripts on the variables are just the numbers 1 through n, we can view a permutation as a rearrangement of the variables. In other words, we can view a permutation σ as changing the product P:

$$P_\sigma = \prod_{i<j}(X_{\sigma(i)} - X_{\sigma(j)}).$$

Example: Suppose $n = 4$ again and σ is the cycle $(1, 3, 4, 2)$. Then

$$P_\sigma = (X_3 - X_1)(X_3 - X_4)(X_3 - X_2)(X_1 - X_4)(X_1 - X_2)(X_4 - X_2).$$

Just like P, the product P_σ will have $n(n-1)/2$ factors. For every pair of variables X_i and X_j, either $X_i - X_j$ or $X_j - X_i$ will be a factor of P_σ. In other words, **either P_σ is P, or it is $-P$.**

Define the function $f\colon S_n \to \{1, -1\}$ by the rule

$$P_\sigma = f(\sigma)P.$$

Certainly the identity function won't change P, so f of the identity function is 1. What about the next simplest type of permutation, transpositions?

Example: Take $n = 4$ again and suppose $\sigma = (1, 3)$. Then

$$P_\sigma = (X_3 - X_2)(X_3 - X_1)(X_3 - X_4)(X_2 - X_1)(X_2 - X_4)(X_1 - X_4) = -P.$$

$$f(\sigma) = -1.$$

Generally speaking, a transposition (a, b) will only affect those factors involving X_a or X_b. Of course, there is one factor involving both of them: $(X_a - X_b)$ (assuming that $a < b$). How many other factors will (a, b) affect? There are $n - 2$ factors involving X_a but not X_b, and $n - 2$ factors involving X_b but not X_a.

Suppose (as we may) that $a < b$. Of those factors involving X_a but not X_b, (a, b) will only change the sign of the factor when the index on the other variable is between a and b. But the same can be said of those factors involving X_b but not X_a. In other words, the combined effect of the transposition (a, b) on all the factors involving just one of X_a or X_b is nothing, since there are an even number of sign changes.

Of course it's easy to see the effect of (a, b) on the factor $(X_a - X_b)$: it changes it to $(X_b - X_a)$.

Theorem: If σ is a transposition, then $f(\sigma) = -1$.

And now for the really crucial step.

Lemma: For any two elements $\sigma, \tau \in S_n$,

$$f(\sigma \circ \tau) = f(\sigma) f(\tau).$$

(not really a) **Proof**: You can, if you like, simply check the four possible cases for $f(\sigma)$ and $f(\tau)$. For example, suppose $f(\sigma) = 1$ and $f(\tau) = -1$. Then applying τ to P first changes it to $-P$. Applying σ to $-P$ won't change it, since σ doesn't change P. So the combined effect of the composition $\sigma \circ \tau$ is to change the sign of P.

Corollary: A permutation $\sigma \in S_n$ can be written as a product of an even number of transpositions if and only if $f(\sigma) = 1$. It can be written as a product of an odd number of transpositions if and only if $f(\sigma) = -1$.

Remark: As mentioned before, an m-cycle can be written as a product of $m-1$ transpositions. Therefore, an m-cycle is even if and only if m is odd.

What are the even permutations in S_3? What are the even permutations in S_4?

Are exactly half of the permutations in S_n even?