Notation and Conventions

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, \ldots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
- $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are viewed as metric spaces using the standard Euclidean distance on $\mathbb{R}^n$.
- $\mathrm{M}_n(\mathbb{R})$ denotes the real vector space of $n \times n$ real matrices, and $\mathrm{M}_n(\mathbb{C})$ the complex vector space of $n \times n$ complex matrices. I denotes the identity matrix in $\mathrm{M}_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{C})$.
- For any $A \in \mathrm{M}_n(\mathbb{C})$, we denote by $\mathrm{tr}(A)$ the trace of A and by $\det(A)$ the determinant of A.
- All rings are associative, with a multiplicative identity.
- For a ring R, $R[x]$ denotes the polynomial ring in one variable over R, and $R^\times$ denotes the multiplicative group of units of R.
- All logarithms are natural logarithms.
- If B is a subset of a set A, we write $A \setminus B$ for the set $\{a \in A \mid a \notin B\}$.

PART A

Answer the following multiple choice questions.

1. Consider the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ defined by

$$a_n = (2^n + 3^n)^{1/n} \text{ and } b_n = \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}}.$$

What is the limit of $\{b_n\}_{n=1}^{\infty}$?

(a) 2.
✓ (b) 3.
(c) 5.
(d) The limit does not exist.

2. Consider the set of continuous functions $f : [0, 1] \to \mathbb{R}$ that satisfy:

$$\int_0^1 f(x)(1 - f(x))\, dx = \frac{1}{4}.$$

Then the cardinality of this set is:

(a) 0.
✓ (b) 1.
(c) 2.
(d) more than 2.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as:

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0, \text{ and} \\ 0, & \text{if } x = 0. \end{cases}$$

Which of the following statements is correct?

✓ (a) f is a surjective function.
(b) f is bounded.
(c) The derivative of f exists and is continuous on $\mathbb{R}$.
(d) $\{x \in \mathbb{R} \mid f(x) = 0\}$ is a finite set.

4. Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing bounded sequence of real numbers such that $\lim_{n\to\infty} a_n = A$. Let $f : [a_1, A] \to \mathbb{R}$ be a continuous function such that for each positive integer i, $f|_{[a_i, a_{i+1}]} : [a_i, a_{i+1}] \to \mathbb{R}$ is either strictly increasing or strictly decreasing. Consider the set

$$B = \{M \in \mathbb{R} \mid \text{ there exist infinitely many } x \in [a_1, A] \text{ such that } f(x) = M\}.$$

Then the cardinality of B is:

(a) necessarily 0.

✓ (b) at most 1.

(c) possibly greater than 1, but finite.

(d) possibly infinite.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies:

$$|f(x) - f(y)| \leq |x - y||\sin(x - y)|, \text{ for all } x, y \in \mathbb{R}.$$

Which of the following statements is correct?

(a) f is continuous but need not be uniformly continuous.

(b) f is uniformly continuous but not necessarily differentiable.

(c) f is differentiable, but its derivative may not be continuous.

✓ (d) f is constant.

6. Let

$$\mathcal{C} = \left\{ f : \mathbb{R} \to \mathbb{R} \mid \ f \text{ is differentiable, and } \lim_{x \to \infty} (2f(x) + f'(x)) = 0 \right\}.$$

Which of the following statements is correct?

(a) For each L with $0 \neq L < \infty$, there exists $f \in \mathcal{C}$ such that $\lim_{x \to \infty} f(x) = L$.

✓ (b) For all $f \in \mathcal{C}$, $\lim_{x \to \infty} f(x) = 0$.

(c) There exists $f \in \mathcal{C}$ such that $\lim_{x \to \infty} f(x)$ does not exist.

(d) There exists $f \in \mathcal{C}$ such that $\lim_{x \to \infty} f(x) = \frac{1}{2}$.

7. Let $f(x) = \dfrac{\log(2 + x)}{\sqrt{1 + x}}$ for $x \geq 0$, and $a_m = \dfrac{1}{m} \int_0^m f(t)\, dt$ for every positive integer m. Then the sequence $\{a_m\}_{m=1}^{\infty}$

(a) diverges to $+\infty$.

(b) has more than one limit point.

(c) converges and satisfies $\lim_{m \to \infty} a_m = \frac{1}{2} \log 2$.

✓ (d) converges and satisfies $\lim_{m \to \infty} a_m = 0$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that:

$$|f(x) - f(y)| \geq \log(1 + |x - y|), \text{ for all } x, y \in \mathbb{R}.$$

Then:

(a) f is injective but not surjective.

(b) f is surjective but not injective.

(c) f is neither injective nor surjective.

✓ (d) f is bijective.

9. What is the greatest integer less than or equal to

$$\sum_{n=1}^{9999} \frac{1}{\sqrt[4]{n}}?$$

✔ (a) 1332
(b) 1352
(c) 1372
(d) 1392

10. Consider the following sentences:

(I) For every connected subset Y of a metric space X, its interior Y° is connected.
(II) For every connected subset Y of a metric space X, its boundary ∂Y is connected.

Which of the following options is correct?

(a) (I) is true, but (II) is false.
(b) (II) is true, but (I) is false.
(c) (I) and (II) are both true.
✔ (d) (I) and (II) are both false.

11. Consider a set $\{A_1, \ldots, A_n\}$ of events, $n > 1$. Suppose that one of the events in $\{A_1, \ldots, A_n\}$ is certain to occur, but that no more than two of them can occur. Suppose that for each $1 \leq r, s \leq n$ such that $r \neq s$, the probability of A_r occurring is p, while the probability of both A_r and A_s occurring is q. Then:

(a) $p \leq 1/n$ and $q \leq 2/n$.
(b) $p \leq 1/n$ and $q \geq 2/n$.
✔ (c) $p \geq 1/n$ and $q \leq 2/n$.
(d) $p \geq 1/n$ and $q \geq 2/n$.

12. Let $\{z_1, z_2, \ldots, z_7\}$ be a set of seven complex numbers with unit modulus. Assume that they form the vertices of a regular heptagon in the complex plane. Define

$$w = \sum_{i<j} z_i z_j.$$

Then:

✔ (a) $w = 0$.
(b) $|w| = \sqrt{7}$.
(c) $|w| = 7$.
(d) $|w| = 1$.

13. Consider $\mathbb{R}^3$ as the space of 3×1 real matrices. The multiplicative group $\mathrm{GL}_3(\mathbb{R})$ of invertible 3×3 real matrices acts on this space by left multiplication. What is the number of orbits for this action?

   (a) 1.
   (b) 2.
   (c) 4.
   (d) ∞.

14. Let V be a finite dimensional vector space over $\mathbb{R}$, and $W \subset V$ a subspace. Then $W \cap T(W) \neq \{0\}$ for every linear automorphism $T : V \to V$ if and only if:

   (a) $W = V$.
   (b) $\dim W < \dfrac{1}{2} \dim V$.
   (c) $\dim W = \dfrac{1}{2} \dim V$.
   (d) $\dim W > \dfrac{1}{2} \dim V$.

15. Let $A \in \mathrm{M}_n(\mathbb{C})$. Then $\begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$ is diagonalizable if and only if:

   (a) $A = 0$.
   (b) $A = I$.
   (c) $n = 2$.
   (d) None of the other three options.

16. Let $T : \mathbb{C} \to \mathbb{R}$ be the map defined by $T(z) = z + \bar{z}$. For a $\mathbb{C}$-vector space V, consider the map

$$\varphi : \{f : V \to \mathbb{C} \mid f \text{ is } \mathbb{C}\text{-linear}\} \to \{g : V \to \mathbb{R} \mid g \text{ is } \mathbb{R}\text{-linear}\},$$

   defined by $\varphi(f) = T \circ f$. Then this map is

   (a) injective, but not surjective.
   (b) surjective, but not injective.
   (c) bijective.
   (d) neither injective nor surjective.

17. Which of the following statements is correct for every linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T^3 - T^2 - T + I = 0$?

   (a) T is invertible as well as diagonalizable.
   (b) T is invertible, but not necessarily diagonalizable.
   (c) T is diagonalizable, but not necessarily invertible.
   (d) None of the other three statements.

18. Let $n \geq 2$. Which of the following statements is true for every $n \times n$ real matrix A of rank one?

✓ (a) There exist matrices $P, Q \in \mathrm{M}_n(\mathbb{R})$ such that all the entries of the matrix PAQ are equal to 1.

(b) There exists an invertible matrix $P \in \mathrm{M}_n(\mathbb{R})$ such that PAP^{-1} is a diagonal matrix.

(c) A has a nonzero eigenvalue.

(d) The vector $(1, 1, \dots, 1) \in \mathbb{R}^n$ is an eigenvector for A.

19. Let m, n be positive integers. Then the greatest common divisor (gcd) of the polynomials $x^m - 1$ and $x^n - 1$ in the ring $\mathbb{C}[x]$ equals

(a) $x^{\min(m,n)} - 1$.

(b) $x - 1$.

✓ (c) $x^{\gcd(m,n)} - 1$.

(d) None of the other three options.

20. Let A_4 denote the group of even permutations of $\{1, 2, 3, 4\}$. Consider the following statements:

(I) There exists a surjective group homomorphism $A_4 \to \mathbb{Z}/4\mathbb{Z}$.

(II) There exists a surjective group homomorphism $A_4 \to \mathbb{Z}/3\mathbb{Z}$.

Which of the following statements is correct?

(a) (I) is true and (II) is false.

✓ (b) (II) is true and (I) is false.

(c) (I) and (II) are both true.

(d) (I) and (II) are both false.

PART B

True/False Questions.

F 1. There exists no monotone function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous at every rational number.

T 2. Let $C([0, 1])$ denote the set of continuous real valued functions on $[0, 1]$, and $\mathbb{R}^{\mathbb{N}}$ the set of all sequences of real numbers. Then there exists an injective map from $C([0, 1])$ to $\mathbb{R}^{\mathbb{N}}$.

T 3. Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of positive real numbers. Then:

$$\limsup_{n\to\infty} \frac{1}{a_n} = \frac{1}{\liminf_{n\to\infty} a_n}.$$

T 4. Let $C([0,1])$ denote the metric space of continuous real valued functions on $[0,1]$ under the supremum metric - i.e., the distance between f and g in $C([0,1])$ equals

$$\sup\{|f(x) - g(x)| \mid x \in [0,1]\}.$$

Let $Q \subset C([0,1])$ be the set of all polynomials in $\mathbb{R}[x]$ in which the coefficient of x^2 is 0. Then Q is dense in $C([0,1])$.

F 5. If X is a metric space such that every continuous function $f : X \to \mathbb{R}$ is uniformly continuous, then X is compact.

T 6. Let X be a metric space, and let $C(X)$ denote the $\mathbb{R}$-vector space of continuous real valued functions on X. Then X is infinite if and only if $\dim_{\mathbb{R}} C(X) = \infty$.

T 7. Let A be a countable union of lines in $\mathbb{R}^3$. Then $\mathbb{R}^3 \setminus A$ is connected.

T 8. An invertible linear map from $\mathbb{R}^2$ to itself takes parallel lines to parallel lines.

F 9. For any matrix C with entries in $\mathbb{C}$, let $m(C)$ denote the minimal polynomial of C, and $p(C)$ its characteristic polynomial. Then for any $n \in \mathbb{N}$, two matrices $A, B \in \mathrm{M}_n(\mathbb{C})$ are similar if and only if $m(A) = m(B)$ and $p(A) = p(B)$.

T 10. Let $A, B \in \mathrm{M}_3(\mathbb{R})$. Then

$$\det(AB - BA) = \frac{\mathrm{tr}[(AB - BA)^3]}{3}.$$

F 11. There exist an integer $r \geq 1$ and a symmetric matrix $A \in \mathrm{M}_r(\mathbb{R})$ such that for all $n \in \mathbb{N}$, we have:

$$2^{\sqrt{n}} \leq |\mathrm{tr}(A^n)| \leq 2020 \cdot 2^{\sqrt{n}}.$$

T 12. The polynomial $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{101}}{101!}$ is irreducible in $\mathbb{Q}[x]$.

F 13. There exists an integer $n > 3$ such that the group of units of the ring $\mathbb{Z}/2^n\mathbb{Z}$ is cyclic.

F 14. For every surjective ring homomorphism $\varphi : R \to S$, we have $\varphi(R^\times) = S^\times$.

F 15. Let G be a finite group and P a p-Sylow subgroup of G, where p is a prime number. Then for every subgroup H of G, $H \cap P$ is a p-Sylow subgroup of H.

T 16. Let G be an abelian group, with identity element e. If

$$\{g \in G \mid g = e \text{ or } g \text{ has infinite order}\}$$

is a subgroup of G, then either all elements of $G \setminus \{e\}$ have infinite order, or all elements of G have finite order.

F 17. There exists a natural number n, with $1 < n \leq 10$, such that x^n and x are conjugate for every element x of S_7, the group of permutations of $\{1, \ldots, 7\}$.

F 18. Every noncommutative ring has at least 10 elements.

T 19. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of elements in $\{0,1\}$ such that for all positive integers n, $\sum_{i=n}^{n+9} a_i$ is divisible by 3. Then there exists a positive integer k such that $a_{n+k} = a_n$ for all positive integers n.

T 20. The interior of any strip bounded by two parallel lines in $\mathbb{R}^2$, of width strictly greater than 1, contains a point with integer coordinates.