

MA 201: Second Order Linear PDE
Canonical Transformation
Lecture - 6

- A second order PDE with two independent variables x and y is given by

$$F(x, y, u, u_x, u_y, u_{xy}, u_{xx}, u_{yy}) = 0. \quad (1)$$

What is the linear form?

- The unknown function $u(x, y)$ satisfies an equation:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0. \quad (2)$$

Facts:

- The expression $Lu \equiv Au_{xx} + Bu_{xy} + Cu_{yy}$ is called the **Principal** part of the equation.
- Classification of such PDEs is based on this principal part.

- At a point (x, y) , the above equation is said to be
 - ▶ Hyperbolic if $B^2(x, y) - 4A(x, y)C(x, y) > 0$
 - ▶ Parabolic if $B^2(x, y) - 4A(x, y)C(x, y) = 0$
 - ▶ Elliptic if $B^2(x, y) - 4A(x, y)C(x, y) < 0$
- Each category relates to specific problems

1. Laplace's Equation

$$u_{xx} + u_{yy} = 0.$$

2. Wave Equation

$$u_{tt} - u_{xx} = 0.$$

3. Heat Equation

$$u_t = u_{xx}.$$

Methods and Techniques for Solving PDEs

- Change of coordinates: A PDE can be changed to an ODE or to an easier PDE by changing the coordinates of the problem.
- Separation of variables: A PDE in n independent variables is reduced to n ODEs.
- Integral transforms: A PDE in n independent variables is reduced to one in $(n - 1)$ independent variables. Hence, a PDE in two variables could be changed to an ODE.
- Numerical Methods

Change of coordinates: Canonical Transformations

- This can be achieved by introducing new coordinates

$$\xi = \xi(x, y), \quad \eta = \eta(x, y).$$

- Compute the partial derivatives

$$\begin{aligned}u_x &= u_\xi \xi_x + u_\eta \eta_x, & u_y &= u_\xi \xi_y + u_\eta \eta_y, \\u_{xx} &= u_{\xi\xi} \xi_x^2 + 2u_{\xi\eta} \xi_x \eta_x + u_{\eta\eta} \eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}, \\u_{yy} &= u_{\xi\xi} \xi_y^2 + 2u_{\xi\eta} \xi_y \eta_y + u_{\eta\eta} \eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}, \\u_{xy} &= u_{\xi\xi} \xi_x \xi_y + u_{\xi\eta} (\xi_x \eta_y + \xi_y \eta_x) + u_{\eta\eta} \eta_x \eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}.\end{aligned}$$

- Substitute these values into the original equation to obtain a new form

$$\tilde{A}u_{\xi\xi} + \tilde{B}u_{\xi\eta} + \tilde{C}u_{\eta\eta} + \tilde{D}u_{\xi} + \tilde{E}u_{\eta} + Fu = G$$

where the new coefficients are as follows

$$\tilde{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2, \quad \tilde{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y,$$

$$\tilde{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2, \quad \tilde{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y,$$

$$\tilde{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y.$$

- Whether the form of PDE remains invariant even after coordinate transformation?

- It can be observed that

$$\begin{pmatrix} 2\tilde{A} & \tilde{B} \\ \tilde{B} & 2\tilde{C} \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} 2A & B \\ B & 2C \end{pmatrix} \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}^t \quad (3)$$

- Taking the determinant on both sides gives

$$\tilde{B}^2 - 4\tilde{A}\tilde{C} = (\xi_x\eta_y - \xi_y\eta_x)^2(B^2 - 4AC) = J^2(B^2 - 4AC) \quad (4)$$

- J is the Jacobian of the transformation and we select the transformation (ξ, η) such that $J \neq 0$.
- Transformation $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ are called canonical transformation or characteristics and the reduced form of the PDE is called canonical form.

Canonical Transformations: Hyperbolic PDE

- Hyperbolic PDE's: $B^2 - 4AC > 0$
 - Natural Choice: $\tilde{A} = 0$ and $\tilde{C} = 0$
 - Leads to Algebraic Equations:

$$A\lambda^2 + B\lambda + C = 0, \quad \lambda_1 = \frac{\xi_x}{\xi_y}, \quad \lambda_2 = \frac{\eta_x}{\eta_y}.$$

► Observations:

- $\xi(x, y) = c$
 - For ξ solve $\frac{dy}{dx} = -\lambda_1(x, y)$
- $$\Rightarrow \quad \xi_x + \xi_y \frac{dy}{dx} = 0,$$
- For η solve $\frac{dy}{dx} = -\lambda_2(x, y)$.
 - Canonical form for hyperbolic equation may be read as

$$u_{\xi\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta).$$

Hyperbolic Equation with Constant Coefficients

- Assume that either $A \neq 0$ or $C \neq 0$.
- What can you say when A and C vanish simultaneously?
- For $A \neq 0$, canonical transformation (ξ, η) takes the form

$$\xi = y + \lambda_1 x, \quad \eta = y + \lambda_2 x, \quad \lambda_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

► For $A \neq 0$ and $C = 0$, we have

$$\xi = y, \quad \eta = y - \frac{B}{A}x.$$

- What are the characteristics when $A = 0$ and $C \neq 0$?

Example

Find the characteristics of the following equation and reduce it to the appropriate standard form and then obtain the general solution:

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0.$$

- We have $\mathbb{D} = B^2 - 4CA = 100 - 36 = 64 > 0$.
- The equation is of hyperbolic type.
- The characteristics are given by

$$\begin{aligned}\xi &= x + \left(\frac{-B + \sqrt{\mathbb{D}}}{2C} \right) y = x - (1/3)y, \\ \eta &= x + \left(\frac{-B - \sqrt{\mathbb{D}}}{2C} \right) y = x - 3y.\end{aligned}$$

- Recall the canonical form

$$\tilde{A}u_{\xi\xi} + \tilde{B}u_{\xi\eta} + \tilde{C}u_{\eta\eta} + \tilde{D}u_{\xi} + \tilde{E}u_{\eta} + Fu = G$$

with

$$\tilde{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2, \quad \tilde{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y,$$

$$\tilde{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2, \quad \tilde{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y,$$

$$\tilde{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y.$$

- Due to the choice of canonical transformation, we have $\tilde{A} = 0 = \tilde{C}$.
- Further, it is easy to see that $\tilde{D} = 0 = \tilde{E} = F$ and $\tilde{B} = -18$.
- Thus, the desired canonical form is

$$u_{\xi\eta} = 0.$$

- What is the general solution?

Canonical Transformations: Parabolic PDE

- Parabolic PDE's: $B^2 - 4AC = 0$

► $A\lambda^2 + B\lambda + C = 0$ has two equal roots

$$\lambda = \lambda_1 = \frac{\xi_x}{\xi_y} = \lambda_2 = \frac{\eta_x}{\eta_y}.$$

► Thus, either $\tilde{A} = 0$ or $\tilde{C} = 0$

► Necessary Requirement $\tilde{B} = 0$

- We obtain only a single characteristic/ canonical transformation as

$$\tilde{A} = 0 \text{ gives } \lambda = \frac{\xi_x}{\xi_y} = -\frac{dy}{dx} \text{ i.e. choice of } \xi \text{ **OR**}$$

$$\tilde{C} = 0 \text{ gives } \lambda = \frac{\eta_x}{\eta_y} = -\frac{dy}{dx} \text{ i.e. choice of } \eta.$$

- Suppose ξ is calculated. What could be the possible choice for η ?
- We select η as any arbitrary function of x and y such that η is independent of ξ i.e.

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0.$$

- For example, if $\xi = x + y$, we may select $\eta = y$ or $\eta = x$.

► Canonical form for Parabolic Equation:

$$\begin{aligned} u_{\eta\eta} &= \phi(\xi, \eta, u, u_\xi, u_\eta), \quad \text{OR} \\ u_{\xi\xi} &= \phi(\xi, \eta, u, u_\xi, u_\eta). \end{aligned}$$

Example

Find the characteristics of the following equation and reduce it to the appropriate standard form and then obtain the general solution:

$$u_{xx} + 4u_{xy} + 4u_{yy} = 0.$$

- We have $B^2 - 4CA = 0$. Hence the equation is parabolic.
- The characteristics are given by

$$\xi = y + \left(\frac{-B + \sqrt{D}}{2A} \right) x = y - 2x, \quad \eta = y.$$

- Recall the canonical form

$$\tilde{A}u_{\xi\xi} + \tilde{B}u_{\xi\eta} + \tilde{C}u_{\eta\eta} + \tilde{D}u_{\xi} + \tilde{E}u_{\eta} + Fu = G$$

with

$$\tilde{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2, \quad \tilde{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y,$$

$$\tilde{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2, \quad \tilde{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y,$$

$$\tilde{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y.$$

- Due to the choice of canonical transformation, we have $\tilde{A} = 0 = \tilde{B}$.
- Further, it is easy to see that $\tilde{D} = 0 = \tilde{E} = F$ and $\tilde{C} = 4$.
- Thus, the desire canonical form is

$$u_{\eta\eta} = 0. \quad \text{What is the general solution?}$$

Canonical Transformations: Elliptic PDE

- Elliptic PDE's: $B^2 - 4AC \leq 0$
 - $A\lambda^2 + B\lambda + C = 0$ has distinct roots λ_1 and λ_2 .
 - Leads to complex conjugate canonical transformation ξ and η .
- Since ξ and η are complex, we introduce new real variables

$$\alpha = \frac{1}{2}(\xi + \eta), \quad \beta = \frac{1}{2i}(\xi - \eta),$$

so that

$$\xi = \alpha + i\beta, \quad \eta = \alpha - i\beta.$$

- Under the transformation $(x, y) \rightarrow (\alpha, \beta)$, the canonical form is given by

$$u_{\alpha\alpha} + u_{\beta\beta} = \Phi(\alpha, \beta, u, u_{\alpha}, u_{\beta}).$$

- Note that method of characteristics is not found suitable for elliptic equations since even after using the transformation, that is, the characteristics (in new variables), the equation gets reduced to Laplace's equation form only.
- In other words: the given equation gets reduced marginally only with two double derivatives still remaining.
- For this reason we will not apply method of characteristics to elliptic equations.

Example

Find the characteristics of the following equation and reduce it to the appropriate standard form and then obtain the general solution:

$$u_{xx} - 4u_{xy} + 4u_{yy} = \cos(2x + y).$$

- The given equation is a parabolic one.
- The characteristics are:

$$\xi = x, \quad \eta = x - \frac{B}{2C}y = x + (1/2)y$$

- The canonical form will be $u_{\xi\xi} = \cos \eta$.
- Integrating partially with respect to ξ :

$$u_{\xi} = \xi \cos \eta + f(\eta).$$

- Again integrating w.r.t. ξ

$$u(\xi, \eta) = \frac{\xi^2}{2} \cos \eta + \xi f(\eta) + g(\eta).$$