$\div$ Solution:-

(1a) $g_n : [\delta, 2\pi-\delta] \to \mathbb{R}$

$$g_n(x) = \frac{1}{n}$$

Then $\left| g_n(x) - 0 \right| = \frac{1}{n} < \varepsilon$ if $n > 1/\varepsilon$ for all

$x \in [\delta, 2\pi-\delta]$

Thus $g_n(x) \to 0$ uniformly on $[\delta, 2\pi-\delta]$ for

$\delta > 0$

Also $g_{n+1}(x) = \frac{1}{n+1} < \frac{1}{n} = g_n(x)$

$\Rightarrow$ $g_{n+1}(x) \leq g_n(x)$ $\forall x \in [\delta, 2\pi-\delta]$

Thus sequence $\langle g_n(x) \rangle$ is uniform convergent to 0 and $\langle g_n \rangle$ is monotonic decreasing for every $x \in [\delta, 2\pi-\delta]$

(1b) $f_n : [\delta, 2\pi-\delta] \to \mathbb{R}$

$$f_n(x) = \cos(nx)$$

Now $S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$

$$S_n(x) = \cos(x) + \cos(2x) + \cdots + \cos(nx)$$

(1b) $f_n : [\delta, 2\pi-\delta] \to \mathbb{R}$

$f_n(x) = \cos(nx)$

Now $S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$

$S_n(x) = \cos(x) + \cos(2x) + \cdots + \cos(nx)$

Now using the following formula

$$\cos(\alpha) + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+(n-1)\beta)$$

$$= \frac{\cos\left(\frac{\alpha + \alpha + (n-1)\beta}{2}\right) \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

we have, $S_n(x) = \dfrac{\cos\left(\frac{x+nx}{2}\right) \sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}$

$\Rightarrow$ $S_n(x) = \dfrac{\cos\frac{(n+1)x}{2} \sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}$, $x \in [\delta, 2\pi-\delta]$

Now let $\phi(x) = \sin\left(\frac{x}{2}\right)$, $x \in [\delta, 2\pi-\delta]$

then $\frac{x}{2} \in [\delta/2, \pi-\delta/2] \subset (0,\pi)$

Thus $\sin\left(\frac{x}{2}\right) > 0 \quad \forall x \in [\delta, 2\pi-\delta]$

So function $g(x) = \dfrac{1}{\sin\left(\frac{x}{2}\right)}$ is continuous on $[\delta, 2\pi-\delta]$, Hence must be bounded on $[\delta, 2\pi-\delta]$

Thus $|g(x)| \leq M_1$ for some $M_1 \in \mathbb{R}^+$

$\Rightarrow$ $\dfrac{1}{\sin\frac{x}{2}} \leq M_1 \quad \forall x \in [\delta, 2\pi-\delta]$

Let $\phi(x) = \sin(\frac{x}{2})$, $x \in [\delta, 2\pi-\delta]$

then $\frac{x}{2} \in [\delta/2, \pi - \delta/2] \subset (0, \pi)$

Thus $\sin(\frac{x}{2}) > 0 \quad \forall x \in [\delta, 2\pi-\delta]$

So function $g(x) = \frac{1}{\sin(\frac{x}{2})}$ is continuous on $[\delta, 2\pi-\delta]$, Hence must be bounded on $[\delta, 2\pi-\delta]$

Thus $|g(x)| \le M_1$ for some $M_1 \in \mathbb{R}^+$

$\Rightarrow \frac{1}{\sin\frac{x}{2}} \le M_1 \quad \forall x \in [\delta, 2\pi-\delta]$

Now $|S_n(x)| = \frac{\left|\cos\frac{(n+1)x}{2}\right| \left|\sin(n\frac{x}{2})\right|}{\sin(\frac{x}{2})} \le \frac{1}{\sin(\frac{x}{2})}$

$\Rightarrow |S_n(x)| \le \frac{1}{\sin(\frac{x}{2})} \le M_1 \quad \forall x \in [\delta, 2\pi-\delta]$

$\Rightarrow |S_n(x)| \le M$ for all $x \in [\delta, 2\pi-\delta]$

So we have two sequences $\langle f_n(x) \rangle$ and $\langle g_n(x) \rangle$ such that

(i) sequence $\langle g_n \rangle \to 0$ uniformly on $[\delta, 2\pi-\delta]$

(ii) sequence $\langle g_n(x) \rangle$ is monotonic for $\forall n \in \mathbb{N}$

(iii) sequence of partial sum of $\langle f_n(x) \rangle$ is uniformly Bounded on $[\delta, 2\pi-\delta]$

Thus from Dirichlet Test, $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n}$ is uniform convergent.