$$G = \left\{ \begin{bmatrix} x & y \\ x & y \end{bmatrix} ; x, y \in \mathbb{R} \text{ s.t. } x+y \neq 0 \right\}$$

① ~~Clousre~~ Property

let $G_1 = \begin{bmatrix} x_1 & y_1 \\ x_1 & y_1 \end{bmatrix}$; $x_1, y_1 \in \mathbb{R}$ & $x_1 + y_1 \neq 0$

$G_2 = \begin{bmatrix} x_2 & y_2 \\ x_2 & y_2 \end{bmatrix}$; $x_1, y_1 \in \mathbb{R}$ & $x_2 + y_2 \neq 0$

now

$$G_1 * G_2 = \begin{bmatrix} x_1 & y_1 \\ x_1 & y_1 \end{bmatrix} \begin{bmatrix} x_2 & y_2 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} x_1x_2 + y_1x_2 & x_1y_2 + y_1y_2 \\ x_1x_2 + y_1x_2 & x_1y_2 + y_1y_2 \end{bmatrix}$$

$$G_1 * G_2 = \begin{bmatrix} x_1x_2 + y_1x_2 & x_1y_2 + y_1y_2 \\ x_1x_2 + y_1x_2 & x_1y_2 + y_1y_2 \end{bmatrix}$$

$\because$ $x_1, y_1 \in \mathbb{R}$ & $x_2, y_2 \in \mathbb{R}$

$\Rightarrow$ $x_1x_2 \in \mathbb{R}$, $y_1y_2 \in \mathbb{R}$, $x_1y_1 \in \mathbb{R}$, $x_1y_2 \in \mathbb{R}$

$\Rightarrow$ $x_1x_2 + y_1x_2 \in \mathbb{R}$; $x_1y_2 + y_1y_2 \in \mathbb{R}$

$\because$ $(x_1 + y_1) \cdot (x_2 + y_2) = x_1x_2 + x_1y_2 + y_1x_2 + y_1y_2$

$\because \quad x_1 + y_1 \neq 0 \quad ; \quad x_2 + y_2 \neq 0$

$\Rightarrow (x_1 + y_1) * (x_2 + y_2) \neq 0$

$\Rightarrow (x_1 x_2 + x_1 y_2) + (x_2 y_1 + y_1 y_2) \neq 0$

$\Rightarrow (x_1 x_2 + x_2 y_1) + (x_1 y_2 + y_1 y_2) \neq 0$

$\Rightarrow \quad G_1 * G_2 \in G$

Closure property hold

(b) <u>Associative</u> :

$\because$ matrix multiplecation follow an associative law

(c) <u>identity element</u>

let $G_1 = \begin{bmatrix} x_1 & y_1 \\ x_1 & y_1 \end{bmatrix} \in G \quad \exists \; H = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$

$x_1 \cdot \frac{x_1}{x_1+y_1} + \frac{x_1 \cdot y_1}{x_1+y_1}$

s.t. $G_1 * H = G_1 \qquad x_1 \left[\frac{x_1+y_1}{x_1+y_1}\right] \therefore x_1$

$\begin{bmatrix} x_1 & y_1 \\ x_1 & y_1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ a & b \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_1 & y_1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} ax_1 + ay_1 & bx_1 + by_1 \\ ax_1 + ay_1 & bx_1 + by_1 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ ax_1 & y_1 \end{bmatrix}$

$$a x_1 + a y_1 = x_1 \Rightarrow a = \frac{x_1}{x_1 + y_1}$$

$$b x_1 + b y_1 = y_1 \Rightarrow b = \frac{y_1}{x_1 + y_1}$$

so

$$H = \begin{bmatrix} \frac{x_1}{x_1+y_1} & \frac{y_1}{x_1+y_1} \\ \frac{x_1}{x_1+y_1} & \frac{y_1}{x_1+y_1} \end{bmatrix}$$

now

$\forall\ G_1 = \begin{bmatrix} x_1 & y_1 \\ x_1 & y_1 \end{bmatrix} \in G\ \exists\ H = \begin{bmatrix} \frac{x_1}{x_1+y_1} & \frac{y_1}{x_1+y_1} \\ \frac{x_1}{x_1+y_1} & \frac{y_1}{x_1+y_1} \end{bmatrix}$

$$G_1 \cdot H = \begin{bmatrix} x_1 & y_1 \\ x_1 & y_1 \end{bmatrix} \cdot \begin{bmatrix} \frac{x_1}{x_1+y_1} & \frac{y_1}{x_1+y_1} \\ \frac{x_1}{x_1+y_1} & \frac{y_1}{x_1+y_1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x_1^2}{x_1+y_1} + \frac{x_1 y_1}{x_1+y_1} & \frac{x_1 y_1 + y_1^2}{x_1+y_1} \\ \frac{x_1^2 + y_1 x_1}{x_1+y_1} & \frac{x_1 y_1 + y_1^2}{x_1+y_1} \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & y_1 \\ x_1 & y_1 \end{bmatrix}$$

so identity element exist

inverse element

for any $G_1 = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_1 \end{bmatrix} \in G \quad \exists \; B = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$

SH $\quad G * B = H$

$$\begin{bmatrix} x_1 & y_1 \\ x_1 & y_1 \end{bmatrix} * \begin{bmatrix} a & b \\ a & b \end{bmatrix} = \begin{bmatrix} \frac{x_1}{x_1+y_1} & \frac{y_1}{x_1+y_1} \\ \frac{x_1}{x_1+y_1} & \frac{y_1}{x_1+y_1} \end{bmatrix}$$

$$\begin{bmatrix} ax_1 + ay_1 & bx_1 + by_1 \\ ax_1 + ay_1 & bx_1 + by_1 \end{bmatrix} = \begin{bmatrix} - & - \\ - & - \end{bmatrix}$$

$$a(x_1+y_1) = \frac{x_1}{(x_1+y_1)}$$

$$a = \frac{x_1}{(x_1+y_1)^2} \quad ; \quad b$$

$$b(x_1+y_1) = \frac{y_1}{(x_1+y_1)} \Rightarrow b = \frac{y_1}{(x_1+y_1)^2}$$

So $a = \dfrac{x_1}{(x_1+y_1)^2}$ $b = \dfrac{y_1}{(x_1+y_1)^2}$

So

for any $G_1 = \begin{bmatrix} x_1 & y_1 \\ x_1 & y_1 \end{bmatrix} \in G$

$$\exists \; B = \begin{bmatrix} \dfrac{x_1}{(x_1+y_1)^2} & \dfrac{y_1}{(x_1+y_1)^2} \\ \dfrac{x_1}{(x_1+y_1)^2} & \dfrac{y_1}{(x_1+y_1)^2} \end{bmatrix} \in G$$

$\Rightarrow$ ~~inverse~~

such that

$$G_1 \cdot B = H$$

So inverse element exist

$\therefore$ $(G, *)$ is a group

Now

let $G_1 = \begin{bmatrix} x_1 & y_1 \\ x_1 & y_1 \end{bmatrix} \in G$

$G_2 = \begin{bmatrix} x_2 & y_2 \\ x_2 & y_2 \end{bmatrix} \in G$

such that

$$G_1 * G_2 = \begin{bmatrix} x_1 & y_1 \\ x_1 & y_1 \end{bmatrix} \begin{bmatrix} x_2 & y_2 \\ x_2 & y_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 x_2 + y_1 x_2 & x_1 y_2 + y_1 y_2 \\ x_1 x_2 + y_1 x_2 & x_1 y_2 + y_1 y_2 \end{bmatrix}$$

and

$$G_2 * G_1 = \begin{bmatrix} x_2 & y_2 \\ x_2 & y_2 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_1 & y_1 \end{bmatrix}$$

$$= \begin{bmatrix} x_2 x_1 + y_2 x_1 & x_2 y_1 + y_2 y_1 \\ x_2 x_1 + y_2 x_1 & x_2 y_1 + y_2 y_1 \end{bmatrix}$$

so $G_1 * G_2 \neq G_2 * G_1$

$\Rightarrow$ $(G_1, *)$ is a group but

not abelian group