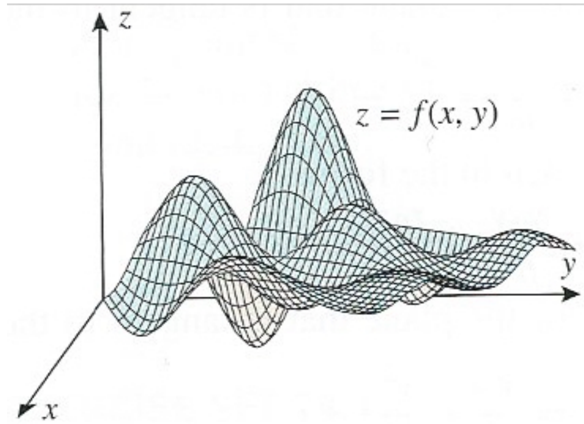
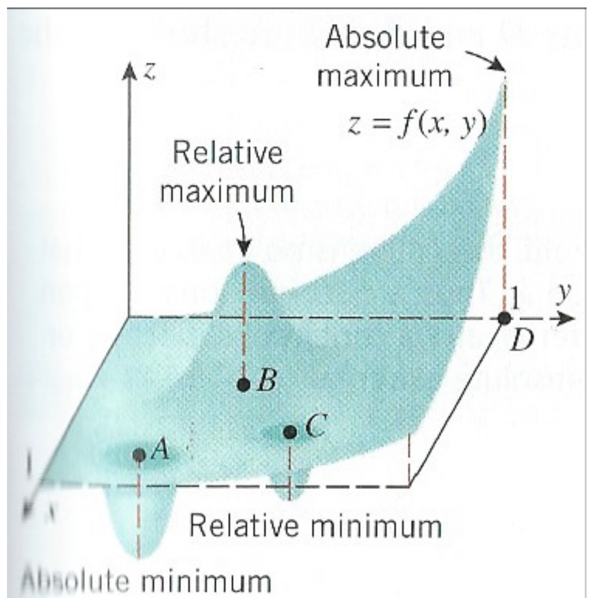
1 Maxima and minima of functions of two variables



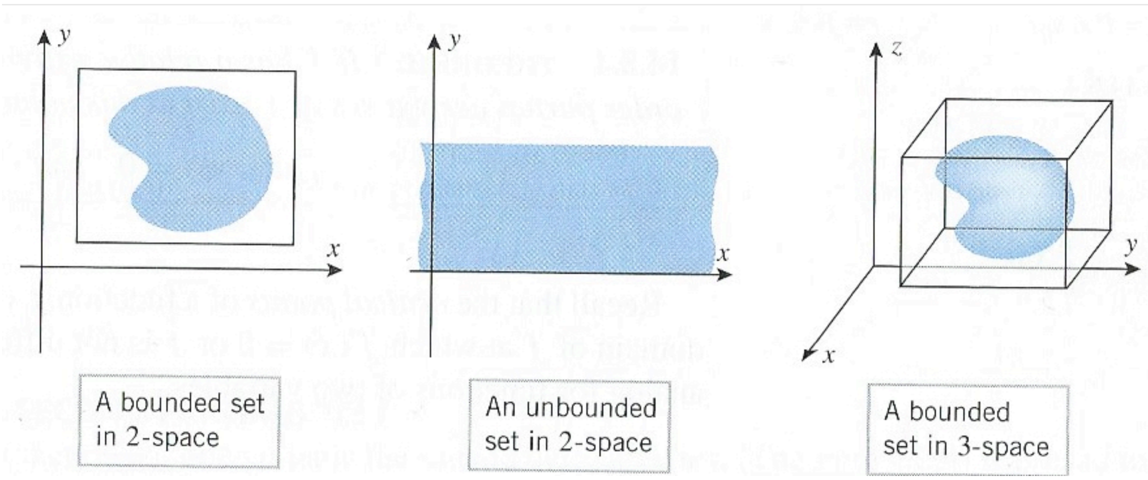
Definition. A function f of two variables is said to have a **relative maximum (minimum)** at a point (a, b) if there is a disc centred at (a, b) such that $f(a, b) \geq f(x, y)$ $(f(a, b) \leq f(x, y))$ for all points (x, y) that lie inside the disc.



A function f is said to have an **absolute maximum (minimum)** at (a, b) if $f(a, b) \geq f(x, y)$ $(f(a, b) \leq f(x, y))$ for all points (x, y) that lie inside in the domain of f.

If f has a relative (absolute) maximum or minimum at (a, b) then we say that f has a **relative (absolute) extremum** at (a, b).

relative $\leftrightarrow$ local



The extreme-value theorem. If $f(x, y)$ is continuous on a closed and bounded set R, then f has both absolute maximum and an absolute minimum on R.

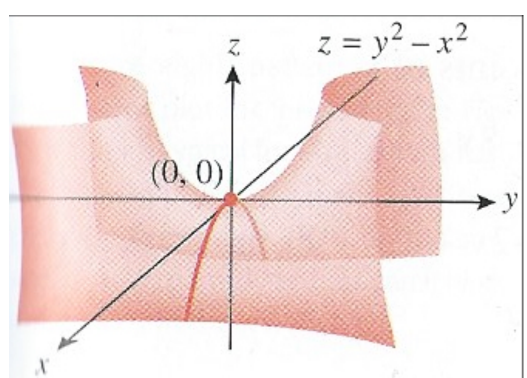
Finding relative extrema

Theorem. If f has a relative extremum at (a, b), and if the first-order derivatives of f exist at this point, then

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0$$

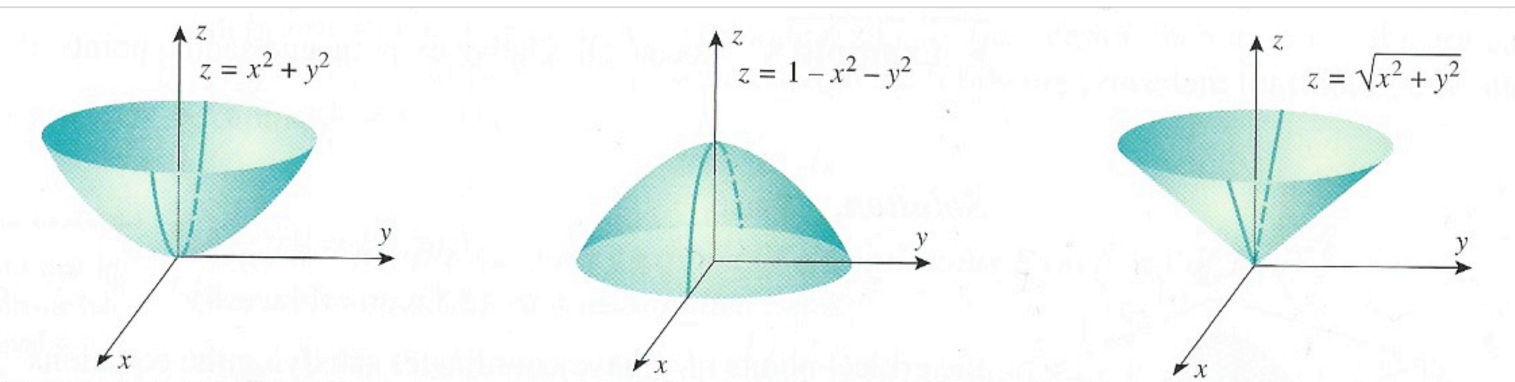
Definition. A point (a, b) in the domain of $f(x, y)$ is called a **critical point** of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one or both partial derivatives do not exist at (a, b).

Example. $f(x, y) = y^2 - x^2$ is a hyperbolic paraboloid.
$f_x = -2x$, $f_y = 2y \Rightarrow (0, 0)$ is critical but it is not a relative extremum.
It is a **saddle point**.



We say that a surface $z = f(x, y)$ has a **saddle point** at (a, b) if there are two distinct vertical planes through this point such that the trace of the surface in one of the planes has a relative maximum at (a, b), and the trace in the other has a relative minimum at (a, b).

Example.



$f(x, y) = x^2 + y^2$ $f_x(0, 0) = f_y(0, 0) = 0$ relative and absolute min at $(0, 0)$	$f(x, y) = 1 - x^2 - y^2$ $f_x(0, 0) = f_y(0, 0) = 0$ relative and absolute max at $(0, 0)$	$f(x, y) = \sqrt{x^2 + y^2}$ $f_x(0, 0)$ and $f_y(0, 0)$ do not exist relative and absolute min at $(0, 0)$

How to determine whether a critical point is a max or min?

The second partials test

Theorem. Let $f(x,y)$ have continuous second-order partial derivatives in some disc centred at a critical point (a,b), and let

$$D = f_{xx}(a,b)f_{yy}(a,b) - \big(f_{xy}(a,b)\big)^2$$

1. If $D > 0$ and $f_{xx}(a,b) > 0$, then f has a relative minimum at (a,b).
2. If $D > 0$ and $f_{xx}(a,b) < 0$, then f has a relative maximum at (a,b).
3. If $D < 0$, then f has a saddle point at(a,b).
4. If $D = 0$, then no conclusion can be drawn.

Example.

$$f(x,y) = x^4 - x^2y + y^2 - 3y + 4$$

How to find the absolute extrema of a continuous function of two variables on a closed and bounded set R?

1. Find the critical points of f that lie in the interior of R.
2. Find all the boundary points at which the absolute extrema can occur.
3. Evaluate $f(x,y)$ at the found points. The largest of these values is the absolute maximum, and the smallest the absolute minimum.

Example.

$$f(x,y) = 3x + 6y - 3xy - 7\,, \quad R \text{ is the triangle } (0,0), (0,3), (5,0)$$

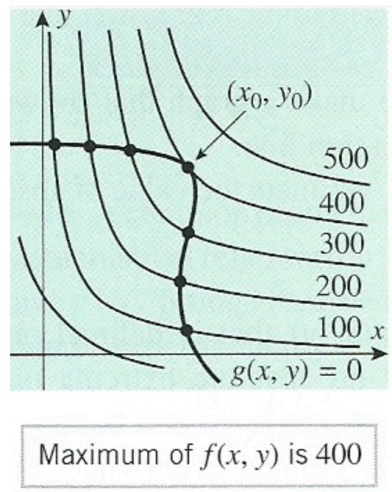
Lagrange multipliers

Extremum problems with constraints:

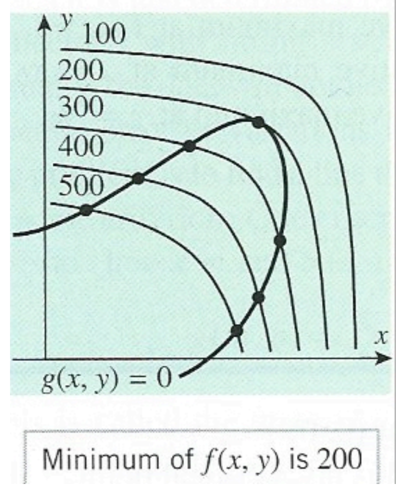
Find max or min of the function $f(x_1, \dots, x_n)$ subject to constraints

$g_\alpha(x_1, \dots, x_n)$, $\alpha = 1, \dots, m$

Consider $f(x, y)$ and $g(x, y) = 0$.
The graph of $g(x, y) = 0$ is a curve.
Consider level curves of f: $f(x, y) = k$.
At (a, b) the curves just touch, and thus have a common tangent line at (a, b). Since $\vec{\nabla} f(a, b)$ is normal to the level curve at (a, b), and $\vec{\nabla} g(a, b)$ is normal to the constraint curve at (a, b), we get $\vec{\nabla} f(a, b) || \vec{\nabla} g(a, b)$



(a)



$$\vec{\nabla} f(a, b) = \lambda \, \vec{\nabla} g(a, b)$$

for some scalar λ called the Lagrange multiplier.

Proof. Parametrise $g(x, y) = 0$.
Then, $f(x, y) = f(x(t), y(t))$ is a function of t and its local extrema are at

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x} x' + \frac{\partial f}{\partial y} y'$$
$$= \vec{\nabla} f \cdot (x' \, \vec{i} + y' \, \vec{j}) = \vec{\nabla} f \cdot \vec{T}$$

Thus, both $\vec{\nabla} f$ and $\vec{\nabla} g$ are $\perp$ to $\vec{T}$.

In general, we introduce a Lagrange multiplier λ_α for each of the constraint g_α, and the equations are

$$\vec{\nabla} f = \sum_{\alpha=1}^{m} \lambda_\alpha \, \vec{\nabla} g_\alpha \, .$$

Example. Find the points on the sphere $x^2 + y^2 + z^2 = 36$ that are closest to and farthest from the point $(1, 2, 2)$.