

MATH 114 QUIZ 3 SOLUTIONS

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1

Find parametric equations for the line tangent to the curve of intersection of the given surfaces at the point $(1, 1, 1)$:

$$\text{Surfaces: } xyz = 1, \quad x^2 + y^2 - z = 1.$$

Solution: Note that the desired tangent line *must* be perpendicular to the normal vectors of both surfaces at the given point. Hence, if we can find the normal vectors of the two surfaces, take their cross product, and then find the parametric equations of a line parallel to the resulting vector, then we'll have solved the problem. With that in mind, remember that the gradient vectors of a function are perpendicular to the function's level surfaces. If we let

$$f(x, y, z) = xyz$$

and

$$g(x, y, z) = x^2 + y^2 - z,$$

then the first surface is the level surface $f(x, y, z) = 1$ and the second is the level surface $g(x, y, z) = 1$. Hence, ∇f will be perpendicular to the first surface and ∇g will be perpendicular to the second surface. Now,

$$\nabla f = (yz)\mathbf{i} + (xz)\mathbf{j} + (xy)\mathbf{k},$$

so

$$\nabla f|_{(1,1,1)} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

On the other hand,

$$\nabla g = (2x)\mathbf{i} + (2y)\mathbf{j} - \mathbf{k},$$

so

$$\nabla g|_{(1,1,1)} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

Therefore, the tangent vector to the curve of intersection of these two surfaces at the point $(1, 1, 1)$ is given by

$$\nabla f|_{(1,1,1)} \times \nabla g|_{(1,1,1)} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 2 & -1 \end{vmatrix} = (-1-2)\mathbf{i} - (-1-2)\mathbf{j} + (2-2)\mathbf{k} = -3\mathbf{i} + 3\mathbf{j}.$$

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Our desired line is parallel to this vector and passes through the point $(1, 1, 1)$, so it is given by the parametric equations

$$\begin{aligned}x(t) &= 1 - 3t \\y(t) &= 1 + 3t \\z(t) &= 1,\end{aligned}$$



Find the local maxima, local minima and saddle points of the function

$$f(x, y) = x^3 + 3xy + y^3.$$

Solution: To find local extrema and saddle points, we must find the critical points of this function, which are those points where $f_x(x, y) = 0 = f_y(x, y)$. Now,

$$\begin{aligned}f_x &= 3x^2 + 3y \\f_y &= 3x + 3y^2.\end{aligned}$$

If $f_x = 0$, then $3x^2 + 3y = 0$, so $3x^2 = -3y$, or $y = -x^2$. If $f_y = 0$, then $3x + 3y^2 = 0$, so $3y^2 = -3x$ or $x = -y^2$. Hence, the critical points of f are given by

$$y = -x^2 = -(-y^2)^2 = -y^4,$$

which is only possible if $y = 0$ or $y = -1$. If $y = 0$, then $x = -y^2 = -0^2 = 0$; if $y = -1$, then $x = -(-1)^2 = -1$. Therefore, the critical points of f are

$$(0, 0), \quad (-1, -1).$$

To determine what sort of critical points these are, we need to use the second derivative test, which requires computing the second derivatives of f :

$$\begin{aligned}f_{xx} &= 6x \\f_{yy} &= 6y \\f_{xy} &= 3\end{aligned}$$

Hence, at $(0, 0)$,

$$f_{xx}(0, 0) \cdot f_{yy}(0, 0) - (f_{xy}(0, 0))^2 = 0 - 9 = -9 < 0,$$

so $(0, 0)$ is a saddle point of f .

On the other hand, at $(-1, -1)$,

$$f_{xx}(-1, -1) \cdot f_{yy}(-1, -1) - (f_{xy}(-1, -1))^2 = (-6) \cdot (-6) - 3^2 = 36 - 9 = 27 > 0,$$

so $(-1, -1)$ is either a minimum or a maximum. To see that $(-1, -1)$ is a local maximum of f , we simply note that

$$f_{xx}(-1, -1) = 6(-1) = -6 < 0.$$

From the above, we conclude that f has a saddle point at $(0, 0)$ and a local maximum of $f(-1, -1) = 1$ at $(-1, -1)$.



2

Find the local extrema of the function $f(x, y) = x^2y$ on the line

$$x + y = 3.$$

Solution: To find the extrema of a function subject to a constraint, we should use Lagrange Multipliers. Let $f(x, y) = x^2y$ and let $g(x, y) = x + y - 3$. Then we have the following setup:

$$\nabla f = \lambda \nabla g$$

$$g(x, y) = 0$$

Now,

$$\nabla f = (2xy)\mathbf{i} + (x^2)\mathbf{j}$$

and

$$\nabla g = (1)\mathbf{i} + (1)\mathbf{j},$$

so the above becomes:

$$(2xy)\mathbf{i} + (x^2)\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j})$$

$$x + y - 3 = 0.$$

Now, we can re-interpret the first equation as a system of two equations, namely:

$$2xy = \lambda$$

$$x^2 = \lambda.$$

In other words, $2xy = x^2$. Now, if $x \neq 0$, then we can divide both sides by x to get

$$2y = x.$$

Substituting this into the equation $g(x, y) = 0$, we see that

$$(2y) + y - 3 = 0.$$

Hence, $3y = 3$, so $y = 1$. Since $x = 2y$, this implies that $x = 2$, so f has a possible local extremum at the point $(2, 1)$.

Remember that we said the above holds only if $x \neq 0$. If $x = 0$, then $x + y = 3$ implies that $y = 3$, so another possible local extremum occurs at $(0, 3)$.

Now, $f(2, 1) = 4$ and it is easy to see that this is a local maximum of f on the line given by $x + y = 3$. On the other hand, $f(0, 3) = 0$ and again it is easy to see that this is a local minimum of f on the line.

