A player tosses a coin and is to score one point for every head turned up and two for every tail. He is to play on until his score reaches or passes $\mathbf{n}$. If $\mathbf{p_n}$ is the chance for attaining exactly $\mathbf{n}$, show that $\mathbf{p_n} = \frac{1}{2}(\mathbf{p_{n-1}} + \mathbf{p_{n-2}})$ and hence find the value of $\mathbf{p_n}$

A $\quad p_n = \frac{1}{3} \times \{2 + \frac{1}{2^n}\}$

B $\quad p_n = \frac{1}{3}\left\{2 + (-1)^n \frac{1}{2^n}\right\}$

C $\quad p_n = \frac{2}{3}\left\{2 + (-1)^n \frac{1}{2^n}\right\}$

D $\quad p_n = \frac{2}{3} \times \{2 + \frac{1}{2^n}\}$

Answer

N points can be scored in the following ways.
Either with n number of heads

$\rightarrow \frac{1}{2^n}$

Or

1 tails and (n-2) number of heads

That is

$T, H, H...(n-2)\text{times}$ ...total n-1 tosses

Internally this could be permuted in $\frac{n-1!}{(n-2)!.1!}$

$= {}^{n-1}C_1$ ways.

Required probability

$= {}^{n-1}C_1 . \frac{1}{2^{n-1}}$

Or

2 tails and (n-4) number of heads

That is

$T, T, H, H...(n-4)\text{times}$ ...total n-2 tosses

Internally this could be permuted in $\frac{n-2!}{(n-4)!.2!}$

$= {}^{n-2}C_2$ ways.

Required probability

$= {}^{n-2}C_2 . \frac{1}{2^{n-2}}$ and so on

Hence

Hence

$\mathbf{P_n}$

$$= \frac{1}{2^n} + {}^{n-1}C_1 \frac{1}{2^{n-1}} + {}^{n-2}C_2 \frac{1}{2^{n-2}} + \ldots.$$

$$= \frac{1}{2^n}[1 + {}^{n-1}C_1.2 + {}^{n-2}C_2 2^2 + {}^{n-3}C_3 2^3 + \ldots]$$

Replacing $\mathbf{n}$ by $\mathbf{n-1}$, we get

$\mathbf{P_{n-1}}$

$$= \frac{1}{2^{n-1}}[1 + {}^{n-2}C_1.2 + {}^{n-3}C_2 2^2 + {}^{n-4}C_3 2^3 + \ldots]$$

$$= \frac{1}{2^n}[2 + {}^{n-2}C_1.2^2 + {}^{n-3}C_2 2^3 + {}^{n-4}C_3 2^4 + \ldots]$$

...(a)

Replacing $\mathbf{n}$ by $\mathbf{n-2}$ we get

$\mathbf{P_{n-2}}$

$$= \frac{1}{2^{n-2}}[1 + {}^{n-3}C_1.2 + {}^{n-4}C_2 2^2 + {}^{n-5}C_3 2^3 + \ldots]$$

$$= \frac{1}{2^n}[4 + {}^{n-3}C_1.2^3 + {}^{n-4}C_2 2^4 + {}^{n-5}C_3 2^5 + \ldots]$$

...(b)

adding a and b, we get

$\mathbf{P_{n-1} + P_{n-2}}$

$$= \frac{1}{2^n}[6 + {}^{n-2}C_1 2^2 + 2^3[{}^{n-3}C_1 + {}^{n-3}C_2] +$$

$$2^4[{}^{n-4}C_2 + {}^{n-4}C_3] + \ldots]$$

$$\frac{1}{2^n}[6 + {}^{n-2}C_1 2^2 + 2^3({}^{n-2}C_2) + 2^4({}^{n-3}C_3) + \ldots]$$

$$= \frac{2}{2^n}[1 + (2 + {}^{n-2}C_1 2^1) + 2^2({}^{n-2}C_2) + 2^3({}^{n-3}C_3) + \ldots]$$

$$= \frac{2}{2^n}[1 + 2({}^{n-2}C_0 + {}^{n-2}C_1) + 2^2({}^{n-2}C_2) + 2^3({}^{n-3}C_3) + \ldots]$$

$$= \frac{2}{2^n}[1 + 2({}^{n-1}C_1) + 2^2({}^{n-2}C_2) + 2^3({}^{n-3}C_3) + \ldots]$$

$$= 2P_n$$

Hence

$$P_{n-1} + P_{n-2} = 2(P_n)$$

Now

$$P_n = \frac{1}{2^n} + {}^{n-1}C_1 \frac{1}{2^{n-1}} + {}^{n-2}C_2 \frac{1}{2^{n-2}} + \ldots.$$

Substituting n=1, we get $P_1 = \frac{1}{2}$

For n=2 $P_2 = \frac{3}{4}$

For n=3 $P_3 = \frac{5}{8}$

for n=4 $P_4 = \frac{11}{16}$

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Hence $\mathbf{P_n = \frac{1}{3}[2 + (-1)^n(\frac{1}{2^n})]}$