$$\int_{-1}^{1} (1-x^2)^n \, dx$$

$1-x^2 \rightarrow$ even function

$$2\int_0^1 (1-x^2)^n \, dx$$

put $x = \sin\theta$

$dx = \cos\theta \, d\theta$

$$2\int_0^{\pi/2} \cos^{(2n+1)}\theta \, d\theta$$

apply gamma function

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{\Gamma\frac{m+1}{2} \, \Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$$

$$\cancel{2}\,\frac{\Gamma(n+1)\,\Gamma\frac{1}{2}}{\cancel{2}\,\Gamma\frac{2n+3}{2}} = \frac{\sqrt{\pi}\; n!}{\Gamma\left(n+\frac{3}{2}\right)}$$

$$= \frac{\sqrt{\pi}\; n!}{\Gamma\left(\left(n+\frac{1}{2}\right)+1\right)} = \frac{\sqrt{\pi}\; n!}{\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)} = \frac{2\sqrt{\pi}\; n!}{(2n+1)\,\Gamma\left(n+\frac{1}{2}\right)}$$

to find $\Gamma(n+1/2)$ use duplication formula

$$\Gamma 2n = 2^{2n-1}\,\frac{\Gamma n\,\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}} \Rightarrow \Gamma\left(n+\frac{1}{2}\right) = \frac{\Gamma 2n}{\Gamma n}\,\frac{\sqrt{\pi}}{2^{2n-1}}$$

$$\frac{(2n)!}{2^{2n}\; n!} \cdot \frac{2\sqrt{\pi}\; n!}{(2n+1)} \cdot \frac{\Gamma n \; 2^{2n-1}}{\Gamma 2n \; \sqrt{\pi}} = \frac{2(n!)}{2n+1}$$