MATH 110: LINEAR ALGEBRA
SPRING 2007/08
PROBLEM SET 2 SOLUTIONS

1. Let V be a vector space over $\mathbb{F}$. Let $\mathbf{w} \in V$ be a fixed non-zero vector and $\mu \in \mathbb{F}$ be a fixed non-zero scalar.

(a) Show that the function $f : \mathbb{F} \to V$ defined by $f(\lambda) = \lambda\mathbf{w}$ is injective.

SOLUTION. If $f(\lambda_1) = f(\lambda_2)$, then $\lambda_1\mathbf{w} = \lambda_2\mathbf{w}$, and so $(\lambda_1 - \lambda_2)\mathbf{w} = \mathbf{0}$. By Theorem **1.3**, we have that $\lambda_1 - \lambda_2 = 0$ since $\mathbf{w} \neq \mathbf{0}$. So $\lambda_1 = \lambda_2$. Hence f is injective.

(b) Show that the function $g : V \to V$ defined by $g(\mathbf{v}) = \mu\mathbf{v}$ is bijective.

SOLUTION. If $g(\mathbf{v}_1) = g(\mathbf{v}_2)$, then $\mu\mathbf{v}_1 = \mu\mathbf{v}_2$, and so $\mu(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}$. By Theorem **1.3**, we have that $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}$ since $\mu \neq 0$. So $\mathbf{v}_1 = \mathbf{v}_2$. Hence g is injective. Since $\mu \neq 0$, it has a multiplicative inverse $\mu^{-1} \in \mathbb{F}$. Given any $\mathbf{v} \in V$ (codomain), the vector $\mu^{-1}\mathbf{v} \in V$ (domain) has the property that $g(\mu^{-1}\mathbf{v}) = \mu(\mu^{-1}\mathbf{v}) = (\mu\mu^{-1})\mathbf{v} = 1\mathbf{v} = \mathbf{v}$. Hence g is surjective.

(c) Show that the function $h : V \to V$ defined by $h(\mathbf{v}) = \mathbf{v} + \mathbf{w}$ is bijective.

SOLUTION. If $h(\mathbf{v}_1) = h(\mathbf{v}_2)$, then $\mathbf{v}_1 + \mathbf{w} = \mathbf{v}_2 + \mathbf{w}$, and adding, $-\mathbf{w}$, the additve inverse of $\mathbf{w}$ to both sides of the equation gives $\mathbf{v}_1 = \mathbf{v}_2$. Hence h is injective. Given any $\mathbf{v} \in V$ (codomain), the vector $\mathbf{v} - \mathbf{w} \in V$ (domain) has the property that $h(\mathbf{v} - \mathbf{w}) = (\mathbf{v} - \mathbf{w}) + \mathbf{w} = \mathbf{v} + (-\mathbf{w} + \mathbf{w}) = \mathbf{v} + \mathbf{0} = \mathbf{v}$. Hence h is surjective.

2. Let W_1 and W_2 be subspaces of a vector space V. The *sum* of W_1 and W_2 is the subset of V defined by

$$W_1 + W_2 = \{\mathbf{w}_1 + \mathbf{w}_2 \in V \mid \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2\}.$$

(a) Prove that $W_1 + W_2$ is a subspace of V.

SOLUTION. Note that $\mathbf{0} = \mathbf{0} + \mathbf{0} \in W_1 + W_2$ and so it is nonempty. Let $\alpha, \beta \in \mathbb{F}$. Let $\mathbf{w}_1 + \mathbf{w}_2, \mathbf{w}_1' + \mathbf{w}_2' \in W_1 + W_2$ where $\mathbf{w}_1, \mathbf{w}_1' \in W_1, \mathbf{w}_2, \mathbf{w}_2' \in W_2$. Since W_1 and W_2 are subspaces, $\alpha\mathbf{w}_1 + \beta\mathbf{w}_1' \in W_1$ and $\alpha\mathbf{w}_2 + \beta\mathbf{w}_2' \in W_2$. Hence

$$\alpha(\mathbf{w}_1 + \mathbf{w}_2) + \beta(\mathbf{w}_1' + \mathbf{w}_2') = (\alpha\mathbf{w}_1 + \beta\mathbf{w}_1') + (\alpha\mathbf{w}_2 + \beta\mathbf{w}_2') \in W_1 + W_2$$

and $W_1 + W_2$ is a subspace by Theorem **1.8**.

(b) Prove that $W_1 + W_2$ is the *smallest* subspace of V containing both W_1 and W_2.

SOLUTION. We need to show that if U is any subspace of V such that

$$W_1 \subseteq U \quad \text{and} \quad W_2 \subseteq U,$$

then

$$W_1 + W_2 \subseteq U.$$

Let $\mathbf{w}_1 + \mathbf{w}_2 \in W_1 + W_2$ where $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$. Since $W_1 \subseteq U$, we must have $\mathbf{w}_1 \in U$. Since $W_2 \subseteq U$, we must have $\mathbf{w}_2 \in U$. And since U is a subspace, we must also have that $\mathbf{w}_1 + \mathbf{w}_2 \in U$. Since our choice of $\mathbf{w}_1 + \mathbf{w}_2 \in W_1 + W_2$ is arbitrary, we may conclude that $W_1 + W_2 \subseteq U$.

(c) Prove that $W_1 \cap W_2$ is the *largest* subspace of V contained in both W_1 and W_2.

SOLUTION. We need to show that if U is any subspace of V such that

$$U \subseteq W_1 \quad \text{and} \quad U \subseteq W_2,$$

then

$$U \subseteq W_1 \cap W_2.$$

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But clearly this is true set theoretically (if $\mathbf{u} \in W_1$ and $\mathbf{u} \in W_2$, then of course $\mathbf{u} \in W_1 \cap W_2$), ie. $W_1 \cap W_2$ is the largest *subset* of V contained in both W_1 and W_2. Since we have shown in the lectures that $W_1 \cap W_2$ is also a subspace, we are done.

3. Let W_1 and W_2 be subspaces of a vector space V. Show that the following statements are equivalent.
(i) $W_1 \cap W_2 = \{\mathbf{0}\}$.
(ii) If $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$ are such that $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{0}$, then $\mathbf{w}_1 = \mathbf{w}_2 = \mathbf{0}$.
(iii) If $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_1' + \mathbf{w}_2'$, where $\mathbf{w}_1, \mathbf{w}_1' \in W_1$ and $\mathbf{w}_2, \mathbf{w}_2' \in W_2$, then $\mathbf{w}_1 = \mathbf{w}_1'$ and $\mathbf{w}_2 = \mathbf{w}_2'$.
If any one of these equivalent conditions holds, then $W_1 + W_2$ is written $W_1 \oplus W_2$ and is called the *direct sum* of W_1 and W_2.
Solution. (i) $\Rightarrow$ (iii): Suppose $\mathbf{w} \in W_1 + W_2$ can be expressed in two (possibly different) ways. Then

$$\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_1' + \mathbf{w}_2'.$$

and hence

$$\mathbf{w}_1 - \mathbf{w}_1' = \mathbf{w}_2' - \mathbf{w}_2. \tag{3.1}$$

Let the vector in (3.1) be denoted by $\mathbf{v}$. Note that $\mathbf{v} = \mathbf{w}_1 - \mathbf{w}_1' \in W_1$ and $\mathbf{v} = \mathbf{w}_2' - \mathbf{w}_2 \in W_2$. So $\mathbf{v} \in W_1 \cap W_2$. Since by (i), $W_1 \cap W_2 = \{\mathbf{0}\}$, this means that $\mathbf{v} = \mathbf{0}$. Therefore, we have

$$\mathbf{w}_1 - \mathbf{w}_1' = \mathbf{0} \quad \text{and} \quad \mathbf{w}_2' - \mathbf{w}_2 = \mathbf{0}$$

and so

$$\mathbf{w}_1 = \mathbf{w}_1' \quad \text{and} \quad \mathbf{w}_2 = \mathbf{w}_2'.$$

(iii) $\Rightarrow$ (ii): If $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{0} = \mathbf{0} + \mathbf{0}$, then $\mathbf{w}_1 = \mathbf{0}$ and $\mathbf{w}_2 = \mathbf{0}$ by (iii).
(ii) $\Rightarrow$ (i): Let $\mathbf{v} \in W_1 \cap W_2$. Since $W_1 \cap W_2$ is a subspace, $-\mathbf{v} \in W_1 \cap W_2$. Since $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$, we have $\mathbf{v} = \mathbf{0}$ by (ii). So $W_1 \cap W_2 = \{\mathbf{0}\}$.

4. (a) State and prove the analogue of the statements in Problem **2** for the direct sum of three or more subspaces.
Solution. The equivalent statements are
(i) $W_i \cap \left(\sum_{j \neq i} W_j\right) = \{\mathbf{0}\}$ for $i = 1, \dots, n$.
(ii) If $\mathbf{w}_1 \in W_1, \dots, \mathbf{w}_n \in W_n$ are such that $\mathbf{w}_1 + \cdots + \mathbf{w}_n = \mathbf{0}$, then $\mathbf{w}_1 = \cdots = \mathbf{w}_n = \mathbf{0}$.
(iii) If $\mathbf{w}_1 + \cdots + \mathbf{w}_n = \mathbf{w}_1' + \cdots + \mathbf{w}_n'$, where $\mathbf{w}_1, \mathbf{w}_1' \in W_1, \dots, \mathbf{w}_n, \mathbf{w}_n' \in W_2$, then $\mathbf{w}_1 = \mathbf{w}_1', \dots, \mathbf{w}_n = \mathbf{w}_n'$.
(i) $\Rightarrow$ (iii): Suppose $\mathbf{w} \in W_1 + \cdots + W_n$ can be expressed in two (possibly different) ways. Then

$$\mathbf{w}_1 + \cdots + \mathbf{w}_n = \mathbf{w}_1' + \cdots + \mathbf{w}_n'. \tag{4.2}$$

and hence

$$\mathbf{w}_1 - \mathbf{w}_1' = (\mathbf{w}_2' - \mathbf{w}_2) + \cdots + (\mathbf{w}_n - \mathbf{w}_n'). \tag{4.3}$$

Let the vector in (4.3) be denoted by $\mathbf{v}$. Note that $\mathbf{v} = \mathbf{w}_1 - \mathbf{w}_1' \in W_1$ and $\mathbf{v} = (\mathbf{w}_2' - \mathbf{w}_2) + \cdots + (\mathbf{w}_n - \mathbf{w}_n') \in W_2 + \cdots + W_n$. So $\mathbf{v} \in W_1 \cap \left(\sum_{j \neq 1} W_j\right)$. Since by (i), $W_1 \cap \left(\sum_{j \neq 1} W_j\right) = \{\mathbf{0}\}$, this means that $\mathbf{v} = \mathbf{0}$. Therefore, we have

$$\mathbf{w}_1 - \mathbf{w}_1' = \mathbf{0} \quad \text{and} \quad (\mathbf{w}_2' - \mathbf{w}_2) + \cdots + (\mathbf{w}_n - \mathbf{w}_n') = \mathbf{0}$$

and so

$$\mathbf{w}_1 = \mathbf{w}_1' \quad \text{and} \quad \mathbf{w}_2 + \cdots + \mathbf{w}_n = \mathbf{w}_2' + \cdots + \mathbf{w}_n.$$

Now repeat the same argument with

$$\mathbf{w}_2 + \cdots + \mathbf{w}_n = \mathbf{w}_2' + \cdots + \mathbf{w}_n'$$

in place of (4.2) to conclude that

$$\mathbf{w}_2 = \mathbf{w}_2' \quad \text{and} \quad \mathbf{w}_3 + \cdots + \mathbf{w}_n = \mathbf{w}_3' + \cdots + \mathbf{w}_n.$$

Upon repeating the argument n times, we obtain

$$\mathbf{w}_1 = \mathbf{w}_1', \dots, \mathbf{w}_n = \mathbf{w}_n'.$$

(iii) $\Rightarrow$ (ii): If $\mathbf{w}_1 + \cdots + \mathbf{w}_n = \mathbf{0} = \mathbf{0} + \cdots + \mathbf{0}$, then $\mathbf{w}_1 = \mathbf{0}, \dots, \mathbf{w}_n = \mathbf{0}$ by (iii).
(ii) $\Rightarrow$ (i): Let $\mathbf{v} \in W_i \cap \left(\sum_{j \neq 1} W_j\right)$. Since $W_i \cap \left(\sum_{j \neq 1} W_j\right)$ is a subspace, $-\mathbf{v} \in W_i \cap \left(\sum_{j \neq 1} W_j\right)$. Since $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$, we have $\mathbf{v} = \mathbf{0}$ by (ii).

(b) Let W_1, W_2, W_3 be subspaces of a vector space V. Suppose

$$W_1 \cap W_2 = W_1 \cap W_3 = W_2 \cap W_3 = \{\mathbf{0}\}.$$

Must $W_1 + W_2 + W_3$ be a direct sum?
Solution. Not necessarily. Let $V = \mathbb{R}^2$. Let $W_1 = \{(a, 0) \in \mathbb{R}^2 \mid a \in \mathbb{R}\}$, $W_2 = \{(0, a) \in \mathbb{R}^2 \mid a \in \mathbb{R}\}$, and $W_3 = \{(a, a) \in \mathbb{R}^2 \mid a \in \mathbb{R}\}$. Note that

$$W_1 \cap W_2 = W_1 \cap W_3 = W_2 \cap W_3 = \{(0, 0)\}$$

but $W_1 + W_2 + W_3$ is not a direct sum since the following three non-zero vectors in W_1, W_2, W_3 add up to $(0, 0)$:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

5. Prove or provide a counter example for the following.

(a) Let

$$V_1 := \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, a, b \in \mathbb{R} \right\},$$
$$V_2 := \left\{ \begin{bmatrix} c & d \\ d & -c \end{bmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, c, d \in \mathbb{R} \right\}.$$

Is it true that

$$\mathbb{R}^{2\times 2} = V_1 \oplus V_2?$$

Solution. Note that

$$\mathbb{R}^{2\times 2} \ni \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x + w & y - z \\ -(y - z) & x + w \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x - w & y + z \\ y + z & -(x + w) \end{bmatrix} \in V_1 + V_2.$$

So

$$\mathbb{R}^{2\times 2} = V_1 + V_2.$$

Let

$$A = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in V_1 \cap V_2.$$

Then we must have $x = w$ since $A \in V_1$ and $y = z$ since $A \in V_2$. Hence

$$A = \begin{bmatrix} x & y \\ y & x \end{bmatrix}.$$

But we must also have $x = -x$ since $A \in V_2$ and $y = -y$ since $A \in V_1$. Hence $x = y = 0$ and

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So $V_1 \cap V_2 = \{\mathbf{0}\}$ and so

$$\mathbb{R}^{2\times 2} = V_1 \oplus V_2.$$

(b) Let

$$W_1 := \{p(x) \in \mathbb{P}_3 \mid p(-x) = p(x) \text{ for all } x \in \mathbb{R}\},$$
$$W_2 := \{p(x) \in \mathbb{P}_3 \mid p(-x) = -p(x) \text{ for all } x \in \mathbb{R}\}.$$

Is it true that

$$\mathbb{P}_3 = W_1 \oplus W_2?$$

Solution. Let $p(x) = a + bx + cx^2 + dx^3$. If $p(-x) = p(x)$, then

$$a - bx + cx^2 - dx^3 = a + bx + cx^2 + dx^3$$

and so $b = d = 0$. If $p(-x) = -p(x)$, then

$$a - bx + cx^2 - dx^3 = -a - bx - cx^2 - dx^3$$

and so $a = c = 0$. Hence

$$W_1 = \{a + cx^2 \mid a, c \in \mathbb{R}\},$$
$$W_2 = \{bx + dx^3 \mid b, d \in \mathbb{R}\}.$$

Clearly

$$\mathbb{P}_3 \ni a + bx + cx^2 + dx^3 = (a + cx^2) + (bx + dx^3) \in W_1 + W_2$$

and so

$$\mathbb{P}_3 = W_1 + W_2.$$

If $p(x) = a + bx + cx^2 + dx^3 \in W_1 \cap W_2$, then $a = c = 0$ and $b = d = 0$. So $W_1 \cap W_2 = \{\mathbf{0}\}$ and so

$$\mathbb{P}_3 = W_1 \oplus W_2.$$