Problem:

$$\int \frac{\cos(x)\sin(x)}{\cos^2(x)+3\cos(x)+2}\,\mathrm{d}x$$

Substitute $u=\cos(x) \longrightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=-\sin(x)$ (steps)

$\longrightarrow \mathrm{d}x=-\frac{1}{\sin(x)}\,\mathrm{d}u$:

$$=-\int \frac{u}{u^2+3u+2}\,\mathrm{d}u$$

... or choose an alternative:

Substitute $\sec(x)$

Now solving:

$$\int \frac{u}{u^2+3u+2}\,\mathrm{d}u$$

Write u as $\frac{1}{2}(2u+3)-\frac{3}{2}$ and split:

$$=\int\left(\frac{2u+3}{2\left(u^2+3u+2\right)}-\frac{3}{2\left(u^2+3u+2\right)}\right)\mathrm{d}u$$

Apply linearity:

$$=\frac{1}{2}\int \frac{2u+3}{u^2+3u+2}\,\mathrm{d}u-\frac{3}{2}\int \frac{1}{u^2+3u+2}\,\mathrm{d}u$$

Now solving:

$$\int \frac{2u+3}{u^2+3u+2}\,\mathrm{d}u$$

Substitute $v = u^2+3u+2 \longrightarrow \frac{\mathrm{d}v}{\mathrm{d}u} = 2u+3$

(steps) $\longrightarrow \mathrm{d}u = \frac{1}{2u+3}\,\mathrm{d}v$:

$$= \int \frac{1}{v}\,\mathrm{d}v$$

This is a standard integral:

$$= \ln(v)$$

Undo substitution $v = u^2+3u+2$:

$$= \ln\left(u^2+3u+2\right)$$

Now solving:

$$\int \frac{1}{u^2+3u+2}\,\mathrm{d}u$$

Factor the denominator:

$$= \int \frac{1}{(u+1)(u+2)}\,\mathrm{d}u$$

Perform partial fraction decomposition:

$$= \int \left(\frac{1}{u+1} - \frac{1}{u+2}\right)\mathrm{d}u$$

Apply linearity:

$$= \int \frac{1}{u+1}\,\mathrm{d}u - \int \frac{1}{u+2}\,\mathrm{d}u$$

Now solving:

$$\int \frac{1}{u+1}\,\mathrm{d}u$$

Substitute $v = u + 1 \longrightarrow \frac{\mathrm{d}v}{\mathrm{d}u} = 1$ (steps) $\longrightarrow$ $\mathrm{d}u = \mathrm{d}v$:

$$= \int \frac{1}{v}\,\mathrm{d}v$$

Use previous result:

$$= \ln(v)$$

Undo substitution $v = u + 1$:

$$= \ln(u+1)$$

Now solving:

$$\int \frac{1}{u+2}\,\mathrm{d}u$$

Substitute $v = u + 2 \longrightarrow \frac{\mathrm{d}v}{\mathrm{d}u} = 1$ (steps) $\longrightarrow$ $\mathrm{d}u = \mathrm{d}v$:

$$= \int \frac{1}{v}\,\mathrm{d}v$$

Use previous result:

$$= \ln(v)$$

Undo substitution $v = u + 2$:

$$= \ln(u+2)$$

$$\int \frac{1}{u+1}\,\mathrm{d}u - \int \frac{1}{u+2}\,\mathrm{d}u$$
$$= \ln(u+1) - \ln(u+2)$$

Plug in solved integrals:

$$\frac{1}{2}\int \frac{2u+3}{u^2+3u+2}\,\mathrm{d}u - \frac{3}{2}\int \frac{1}{u^2+3u+2}\,\mathrm{d}u$$
$$= \frac{\ln\left(u^2+3u+2\right)}{2} + \frac{3\ln(u+2)}{2} - \frac{3\ln(u+1)}{2}$$

Plug in solved integrals:

$$-\int \frac{u}{u^2+3u+2}\,\mathrm{d}u$$

$$= -\frac{\ln\left(u^2+3u+2\right)}{2} - \frac{3\ln(u+2)}{2} + \frac{3\ln(u+1)}{2}$$

Undo substitution $u = \cos(x)$:

$$= -\frac{\ln\left(\cos^2(x)+3\cos(x)+2\right)}{2} - \frac{3\ln(\cos(x)+2)}{2} + \frac{3\ln(\cos(x)+1)}{2}$$

The problem is solved:

$$\int \frac{\cos(x)\sin(x)}{\cos^2(x)+3\cos(x)+2}\,\mathrm{d}x$$

$$= -\frac{\ln\left(\cos^2(x)+3\cos(x)+2\right)}{2} - \frac{3\ln(\cos(x)+2)}{2} + \frac{3\ln(\cos(x)+1)}{2} + C$$