

$$\sum_{n=1}^{\infty} \left[\frac{1}{n} - \sin\left(\frac{1}{n}\right) \right]^{\alpha}$$

$$\therefore \text{Expansion of } \sin\left(\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{3!}n^{-3} + \frac{1}{5!}n^{-5} - \frac{1}{7!}n^{-7} + \frac{1}{9!}n^{-9} - \dots$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{n} - \left(\frac{1}{n} - \frac{1}{3!}n^{-3} + \frac{1}{5!}n^{-5} - \frac{1}{7!}n^{-7} + \frac{1}{9!}n^{-9} - \dots \right) \right]^{\alpha}$$

$$= \sum_{n=1}^{\infty} \left[\cancel{\frac{1}{n}} - \cancel{\frac{1}{n}} + \frac{1}{3!}n^{-3} + \frac{1}{5!}n^{-5} + \frac{1}{7!}n^{-7} - \frac{1}{9!}n^{-9} + \frac{1}{11!}n^{-11} - \dots \right]^{\alpha}$$

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n^{3\alpha}} \left[\frac{1}{3!} - \frac{1}{5!}n^{-2} + \frac{1}{7!}n^{-4} - \dots \right]^{\alpha}$$

Now let an auxiliary equation $\sum v_n = \sum \frac{1}{n^{3\alpha}}$ such that by limit comparison test

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^{3\alpha}} \left[\frac{1}{3!} - \frac{1}{5!}n^{-2} + \dots \right]}{\frac{1}{n^{3\alpha}}}$$

$$= \frac{1}{3!}$$

Hence Both series will be either convergent or divergent simultaneously, so

Now $\sum v_n = \sum \frac{1}{n^{3\alpha}}$ will be convergent if

$$3\alpha > 1$$

$$\alpha > \frac{1}{3}$$

By p test
 $\sum \frac{1}{n^p}$ is convergent if $p > 1$
 divergent if $p \leq 1$

So $\sum \left[\frac{1}{n} - \sin\left(\frac{1}{n}\right) \right]^{\alpha}$ will convergent for $\alpha > \frac{1}{3}$

option A is correct

$$\sum_{n=1}^{\infty} \left[\frac{1}{n} - \sin\left(\frac{1}{n}\right) \right]^{\alpha}$$

$$\begin{aligned} \therefore \text{Expansion of } \sin\left(\frac{1}{n}\right) &= \frac{1}{n} - \frac{1}{3!}n^{-3} + \frac{1}{5!}n^{-5} - \frac{1}{7!}n^{-7} + \frac{1}{9!}n^{-9} - \dots \\ &= \sum_{n=1}^{\infty} \left[\frac{1}{n} - \left(\frac{1}{n} - \frac{1}{3!}n^{-3} + \frac{1}{5!}n^{-5} - \frac{1}{7!}n^{-7} + \frac{1}{9!}n^{-9} - \frac{1}{11!}n^{-11} + \dots \right) \right]^{\alpha} \\ &= \sum_{n=1}^{\infty} \left[\cancel{\frac{1}{n}} - \cancel{\frac{1}{n}} + \frac{1}{3!}n^{-3} + \frac{1}{5!}n^{-5} + \frac{1}{7!}n^{-7} - \frac{1}{9!}n^{-9} + \frac{1}{11!}n^{-11} - \dots \right]^{\alpha} \end{aligned}$$

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n^{3\alpha}} \left[\frac{1}{3!} - \frac{1}{5!}n^{-2} + \frac{1}{7!}n^{-4} - \dots \right]^{\alpha}$$

Now let an auxiliary equation $\sum v_n = \sum \frac{1}{n^{3\alpha}}$ such that by limit comparison test

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^{3\alpha}} \left[\frac{1}{3!} - \frac{1}{5!}n^{-2} + \dots \right]}{\frac{1}{n^{3\alpha}}}$$

$$= \frac{1}{3!}$$

Hence Both series will be either convergent or divergent simultaneously, so

Now $\sum v_n = \sum \frac{1}{n^{3\alpha}}$ will be convergent if

$$\begin{aligned} 3\alpha &> 1 \\ \alpha &> \frac{1}{3} \end{aligned} \quad \left[\begin{array}{l} \therefore \text{By } p \text{ test} \\ \sum \frac{1}{n^p} \text{ is convergent if } p > 1 \\ \text{divergent if } p \leq 1 \end{array} \right]$$

So $\sum \left[\frac{1}{n} - \sin\left(\frac{1}{n}\right) \right]^{\alpha}$ will be convergent for $\alpha > \frac{1}{3}$

option A is correct