The Radius of Convergence Formulas

Theorem: Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$.

a. Suppose that the limit $\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}$ exists or is ∞. Then the radius of convergence R of the power series is given by

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}.$$

b. Suppose that the limit $\lim_{n\to\infty} |c_n|^{1/n}$ exists or is ∞. Then the radius of convergence R of the power series is given by

$$\frac{1}{R} = \lim_{n\to\infty} |c_n|^{1/n}.$$

Proof: We will prove part **(a)**. The proof of part **(b)** is similar. Suppose that $\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \ell$ where $0 \le \ell \le \infty$. We apply the Ratio Test to the series $\sum_{n=0}^{\infty} c_n(x-a)^n$. Since $a_n = c_n(x-a)^n$, we obtain

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|c_{n+1}(x-a)^{n+1}|}{|c_n(x-a)^n|} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} \cdot |x-a| = \ell|x-a| .$$

By the Ratio Test, the series converges absolutely if $\ell|x-a| < 1$, that is if $|x-a| < 1/\ell$; and diverges if $\ell|x-a| > 1$, that is if $|x-a| > 1/\ell$. It follows that $R = 1/\ell$ by the definition of the radius of convergence. □

Remark: Note that the series has the form $\sum_{n=0}^{\infty} c_n(x-a)^n$ and c_n is the coefficient of $(x-a)^n$. This formulas cannot be applied to series that does not come in this form.

Examples: Find the radius of convergence of each of the following power series.

1. $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2^n+1)(n^2+1)}$

From

$$c_n = \frac{(-1)^n}{(2^n+1)(n^2+1)}$$

we obtain

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{1/((2^{n+1}+1)((n+1)^2+1))}{1/((2^n+1)(n^2+1))} = \lim_{n\to\infty} \left(\frac{1+2^{-n}}{2+2^{-n}} \cdot \frac{n^2+1}{(n+1)^2+1} \right) = \frac{1}{2},$$

and therefore $R = 2$. Or we can use the other formula:

$$\frac{1}{R} = \lim_{n\to\infty} |c_n|^{1/n} = \lim_{n\to\infty} \left(\frac{1}{(2^n+1)(n^2+1)} \right)^{1/n} = \lim_{n\to\infty} \frac{1}{2 \cdot (1+2^{-n})^{1/n} \cdot (n^{1/n})^2 \cdot (1+n^{-2})^{1/n}} = \frac{1}{2},$$

and again $R = 2$.

2. $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2^n+1)(n^2+1)}$

Note that we cannot apply the formulas in the Theorem in this case. Instead we will use the nth Root Test directly.

$$a_n = (-1)^n \frac{x^{2n+1}}{(2^n+1)(n^2+1)}$$

gives

$$L = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{|x|^{2n+1}}{(2^n+1)(n^2+1)}\right)^{1/n} = \lim_{n\to\infty} \frac{|x|^2 \cdot |x|^{1/n}}{2\cdot(1+2^{-n})^{1/n}\cdot(n^{1/n})^2\cdot(1+n^{-2})^{1/n}} = \frac{|x|}{2}^2 .$$

By the nth Root Test, the series converges absolutely if $|x|^2/2 < 1$, that is if $|x| < \sqrt{2}$; and diverges if $|x|^2 > 2$, that is if $|x| > \sqrt{2}$. Therefore, by the definition of radius of convergence, we have $R = \sqrt{2}$.

Another way of solving this problem is to observe that

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2^n+1)(n^2+1)}$$

has the same radius of convergence as

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2^n+1)(n^2+1)} ,$$

and then use the change of variable $u = x^2$ to write the latter series as

$$\sum_{n=0}^{\infty}(-1)^n \frac{u^n}{(2^n+1)(n^2+1)} .$$

Now this is the series in *Example 1* whose radius we found as $R_u = 2$. This means that

$$\sum_{n=0}^{\infty}(-1)^n \frac{u^n}{(2^n+1)(n^2+1)}$$

converges absolutely for $|u| < 2$, and diverges for $|u| > 2$. Therefore

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2^n+1)(n^2+1)}$$

converges absolutely for $|x^2| < 2$, and diverges for $|x^2| > 2$. In other words, its radius of convergence is $R = \sqrt{2}$. This implies that the radius of convergence of the original series

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2^n+1)(n^2+1)}$$

is also $R = \sqrt{2}$.