$$(x^4 D^4 + 6x^3 D^3 + 9x^2 D^2 + 3xD + 1)\,y = (1+\log x)^2 \quad (1)$$

$$D = \frac{d}{dx}$$

Now put $x = e^z$ so $z = \log x$ — (2)

$$\frac{dx}{dz} = e^z \quad ; \quad \frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{1}{x}$$

$$\Rightarrow \quad x\frac{dy}{dx} = \frac{dy}{dz} \quad \Rightarrow (xD)y = D'y \quad \text{where } D' = \frac{d}{dz} \quad (3)$$

Again $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\underbrace{\frac{1}{x}}_{I}\cdot\underbrace{\frac{dy}{dz}}_{II}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\right)\frac{dy}{dz} + \frac{1}{x}\cdot\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}$$

$$\because \frac{dz}{dx} = \frac{1}{x}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\cdot\frac{d^2y}{dz^2}$$

$$x^2\frac{d^2y}{dx^2} = -\frac{dy}{dz} + \frac{d^2y}{dz^2}$$

$$\Rightarrow (x^2D^2)y = (D'^2 - D')y \quad \text{or}$$

$x^2 D^2 = D'(D'-1)$ — (4)

similarly $x^3 D^3 = D'(D'-1)(D'-2)$ — (5)

$x^4 D' = D'(D'-1)(D'-2)(D'-3)$ — (6)

from eqn (1) (2), (3), (4) (5) and (6) we get

$[D'(D'-1)(D'-2)(D'-3) + 6D'(D'-1)(D'-2) + 9D'(D'-1) + 3D'+1]y = (1+z)^2$

$\Rightarrow [D'(D'-1)(D'-2)[D'-3+6] + 9D'(D'-1) + 3D'+1]y = (1+z)^2$

$\Rightarrow [D'(D'-1)(D'-2)(D'+3) + 9D'(D'-1) + 3D'+1]y = (1+z)^2$

$\Rightarrow [D'(D'-1)\{(D'-2)(D'+3) + 9\} + 3D'+1]y = (1+z)^2$

$\Rightarrow [D'(D'-1)\{D'^2 + D' - 6 + 9\} + 3D'+1]y = (1+z)^2$

$\Rightarrow [D'(D'-1)(D'^2 + D' + 3) + 3D'+1]y = (1+z)^2$

$[D'(D'-1)D'(D'+1) + 3D'(D'-1) + 3D'+1]y = (1+z)^2$

$\Rightarrow (D'^2(D'^2-1) + 3D'^2 - \cancel{3D'} + \cancel{3D'} + 1]y = (1+z)^2$

$\Rightarrow [D'^4 + 2D'^2 + 1]y = (1+z)^2$

$\Rightarrow (D'^2+1)^2 y = (1+z)^2$

To find C.F. its auxiliary equation is ~~equt~~ given as

$(m^2+1)^2 = 0$

$\Rightarrow (m^2+1)(m^2+1) = 0$

$m^2+1 = 0$ & $m^2+1 = 0$

$m = \pm i$; $m = \pm i$

roots are purely imaginary and repeated

hence

C.F $= (C_1 + C_2 z)\cos z + (C_3 + C_4 z)\sin z$

$\because$ Here we are using independent variable z because we already converted the eqn ① into $D = \frac{d}{dz}$ form. so independent variable is now z instead x

and $\because$ $z = \log x$ (from eqn 2)

C.F $= (C_1 + C_2 (\log x))\cos(\log x) + (C_3 + C_4 (\log x))\sin(\log x)$