Que $(D^2+a^2)y = \sec$

Auxiliary equation

$$m^2 + a^2 = 0$$

$$\Rightarrow \quad m = \pm a i$$

C.F. $= (c_1 \cos ax + c_2 \sin ax)$

P.I. $= \dfrac{\sec ax}{D^2 + a^2}$

$\because$ we know that $(x+ia)(x-ia) = x^2 - iax + iax - i^2a^2$

$$= x^2 - i^2 a^2 \qquad \text{where } i = \sqrt{-1}, \quad i^2 = -1$$

$$= x^2 + a^2$$

$$(x+ia)(x-ia) = x^2 + a^2$$

So $(D^2+a^2) = (D+ia)(D-ia)$

$$= \frac{\sec ax}{(D+ia)(D-ia)}$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right] \sec ax$$

$$= \frac{1}{2ia}\left[\frac{\sec ax}{D-ia} - \frac{\sec ax}{D+ia}\right]$$

$$= \frac{1}{2ia}\left[e^{iax}\int e^{-iax} \sec ax \, dx - e^{-iax}\int e^{ax} \sec ax \, dx\right]$$

$\because$

we know that $\dfrac{f(x)}{D+a} = e^{-ax}\int e^{ax} f(ax)\,dx$

$\because e^{iax} = \cos ax + i \sin ax$

$\& \; e^{-iax} = \cos ax - i \sin ax$

$$\text{so } = \frac{1}{2ia}\left[e^{iax}\int e^{-iax}\sec ax\,dx - e^{-iax}\int e^{iax}\sec ax\,dx\right]$$

$$= \frac{1}{2ai}\left[e^{iax}\int(\cos ax - i\sin ax)\sec ax\,dx - e^{-iax}\int(\cos ax + i\sin ax)\sec ax\,dx\right]$$

$\because \sec\theta = \frac{1}{\cos\theta}$

$$= \frac{1}{2ia}\left[e^{iax}\int(1 - i\tan ax)\,dx - e^{-iax}\int(1 + i\tan ax)\,dx\right]$$

$$= \frac{1}{2ia}\left[e^{iax}\left\{\int dx - i\int\tan ax\,dx\right\} - e^{-iax}\left\{\int dx + i\int\tan ax\,dx\right\}\right]$$

$$\because \left[\begin{aligned}\int\tan x\,dx &= \log\sec x\\ &= -\log\cos x\end{aligned}\right]$$

$$= \frac{1}{2ia}\left[e^{iax}\left\{x + \frac{i}{a}\log\cos ax\right\} - e^{-iax}\left\{x - \frac{i}{a}\log\cos ax\right\}\right]$$

$$= \frac{1}{2ia}\left[(\cos ax + i\sin ax)\left\{x + \frac{i}{a}\log\cos ax\right\} - (\cos ax - i\sin ax)\left\{x - \frac{i}{a}\log\cos ax\right\}\right]$$

$$= \frac{1}{2ia}\Big[x\cos ax + i x\sin ax + \frac{i}{a}\cos ax\log\cos ax + \frac{i^2}{a}\sin ax\log\cos ax$$

$$- x\cos ax + i x\sin ax + \frac{i}{a}\cos ax\log\cos ax - \frac{i^2}{a}\sin ax\log\cos ax\Big]$$

$$= \frac{1}{2ai}\left[\frac{2i}{a}\cos ax\log\cos ax + 2ix\sin ax\right]$$

$$P.I. = \frac{1}{a^2} \cos ax \log \cos ax + \frac{x}{a} \sin ax$$

hence the general solution is given as

$$y(x) = C.F. + P.I.$$