Let (1) $S = \sum_{n=1}^{\infty} \frac{n^2}{2^n}$

This can be written

(2) $S = \sum_{n=0}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} \frac{(n-1)^2}{2^{n-1}} = 2\sum_{n=1}^{\infty} \frac{(n-1)^2}{2^n}$

From (1) and (2) we get

S – ½ S = $\sum_{n=1}^{\infty} \frac{n^2}{2^n} - \sum_{n=1}^{\infty} \frac{(n-1)^2}{2^n}$

= $\sum_{n=1}^{\infty} \frac{n^2-(n-1)^2}{2^n}$ = $\sum_{n=1}^{\infty} \frac{n^2-(n^2-2n+1)}{2^n}$

(3) ½ S = $\sum_{n=1}^{\infty} \frac{2n-1}{2^n}$ or S = 2 $\sum_{n=1}^{\infty} \frac{2n-1}{2^n}$

If we extract the first term in the series, we get:

S = 2 (½ + $\sum_{n=2}^{\infty} \frac{2n-1}{2^n}$) = 1 + 2 $\sum_{n=1}^{\infty} \frac{2n+1}{2^{n+1}}$) (moving from n to n-1)

(4) S = 1 + $\sum_{n=1}^{\infty} \frac{2n+1}{2^n}$)

From (4) and (3) we have:

S = 2S – S

= 2(1 + $\sum_{n=1}^{\infty} \frac{2n+1}{2^n}$) – 2 $\sum_{n=1}^{\infty} \frac{2n-1}{2^n}$

= 2 + 2 $\sum_{n=1}^{\infty} \frac{2n+1}{2^n}$ –2 $\sum_{n=1}^{\infty} \frac{2n-1}{2^n}$

= 2 + 2 $\sum_{n=1}^{\infty} \frac{2n}{2^n}$ + 2 $\sum_{n=1}^{\infty} \frac{1}{2^n}$ –2 $\sum_{n=1}^{\infty} \frac{2n}{2^n}$ + 2 $\sum_{n=1}^{\infty} \frac{1}{2^n}$

This gives

S = 2 + + 2 $\sum_{n=1}^{\infty} \frac{1}{2^n}$ + 2 $\sum_{n=1}^{\infty} \frac{1}{2^n}$

= 2 + 2 + 2 = 6

So S = 6