POWER SEMIGROUPS: RESULTS AND PROBLEMS

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ABSTRACT

This is a survey of work on the algebraic theory of the power operator on pseudovarieties of semigroups and monoids. Besides exploring connections with other operators, various problems and partial solutions are presented. Recent generalizations of the power operator, particularly the one resulting from looking only at group-pointlike subsets of finite monoids, are also considered.

1. Introduction

For any algebraic structure S, one may define on the power set $\mathscr{P}(S)$ a similar algebraic structure by letting the result of applying an operation to a bunch of subsets of S be the set of all operation values which can be obtained by replacing each set by one of its elements. The question that is then naturally raised concerns which properties are preserved in this construction. In the context of Universal Algebra, this question has been considered by looking at equational properties or, equivalently, by defining the power operator on varieties and looking for fixed points. The answer turns out to be rather simple (cf. [26, 55]): the equational properties which are preserved are precisely those in which both sides of the equation have no repeated variables and involve the same variables; moreover, the power operator on varieties is an idempotent.

Now, if we are only interested in finite algebras, Birkhoff's variety theory is no longer appropriate. In the context of semigroup theory and its applications, the convenient corresponding notion is that of a *pseudovariety* [29] and the appropriate defining axioms are known as pseudoidentities [52, 11]. Moreover, the reason why pseudovarieties were first considered was that they naturally came up in the applications involving syntactical characterizations of classes of rational languages. Eilenberg [28] provided an abstract framework for such applications in the form of a one-to-one correspondence between the so-called *varieties of languages* and pseudovarieties of semigroups. In turn, operators on varieties of languages therefore correspond to operators on pseudovarieties of semigroups. The motivation for the first studies of the power operator on pseudovarieties came precisely from its discovery as the pseudovariety operator associated with some natural operators on varieties of languages [43].

*This work was supported, in part, by F.C.T. through the *Centro de Matemática da Universidade do Porto*, the project Praxis XXI/2/2.1/MAT/63/94, and the Department of Mathematics of the University of Tasmania.

The situation for the power operator on pseudovarieties turns out to be much more complicated than its variety version. Here, the operator is not an idempotent, although its iteration stops in 3 steps. While the fixed points have been characterized, the pseudoidentities defining them are not as simple as in the variety case and the image of the power operator remains largely unknown. Also, some specific calculations involving the power operator have turned out to relate very closely to other crucial problems in the theory of finite semigroups.

This paper is a survey of the work done since the early 1980's on the power operator. It explores some of the connections with other operators on pseudovarieties and contains many open problems. Some generalizations of the power operator suggested by Ash's work [21] are also considered.

The reader is assumed to have a basic background in finite semigroup theory, say as presented in [46]. For more comprehensive treatments of the theory, see [28, 11].

2. Examples

We consider in this section some elementary examples leading nevertheless to some important observations. We denote by $\mathscr{P}(S)$ (respectively $\mathscr{P}'(S)$) the semigroup of all (respectively nonempty) subsets of a semigroup S.

2.1. Aperiodic Brandt semigroups

Let M_n denote the multiplicative semigroup of all $n \times n$ matrices with entries in the Boolean ring $\{0, 1\}$. Such matrices may be viewed as binary relations on an n-element set so that matrix multiplication corresponds to usual relation composition. In particular, every finite semigroup with n elements embeds in M_{n+1}.

Let B_n be the subsemigroup of M_n consisting of all matrices with, at most, one nonzero entry. When $m < n$, by completing an $m \times m$ matrix to an $n \times n$ matrix by adding rows and columns of zeros, we see that B_m embeds in B_n. On the other hand, the Kronecker product defines an onto homomorphism $B_m \times B_n \to B_{mn}$. Finally, the mapping $\mathscr{P}'(B_n) \to M_n$ sending a nonempty set X of matrices to the sum of the elements of X (i.e., the matrix having an entry 1 precisely at those entries where some element of X has value 1) is an onto homomorphism.

So, if we say that a semigroup S *divides* a semigroup T (written $S \prec T$) if S is a homomorphic image of a subsemigroup of T, and we write $S^{(n)}$ for the direct product of n copies of S, we have established the following result.

Proposition 2.1. *Every finite semigroup divides some* $\mathscr{P}'(B_2^{(n)})$.$\square$

2.2. Groups

Let S be a semigroup and suppose D is one of its regular $\mathscr{D}$-classes. Let $\sim$ be the least equivalence relation on the set of group elements of D which identifies two elements if they are either $\mathscr{R}$ or $\mathscr{L}$-equivalent. A *block* of D is a Rees quotient of the subsemigroup of S generated by a $\sim$-class, modulo the ideal consisting of the elements which do not lie in D. The *blocks* of S are the blocks of its regular $\mathscr{D}$-classes and are therefore completely [0]-simple semigroups.

Let G be a finite group and consider the semigroup $\mathscr{P}'(G)$. Then the following facts are easily established:

- the idempotents of $\mathscr{P}'(G)$ are the subgroups of G;
- two elements A and B of $\mathscr{P}'(G)$ are $\mathscr{R}$ (respectively $\mathscr{L}$)-equivalent if and only if there exists $x \in G$ such that $Ax = B$ (respectively $xA = B$);
- let $H \in \mathscr{P}'(G)$ be an idempotent; the $\mathscr{R}$ (respectively $\mathscr{L}$)-class of H is the set G/H of its right cosets (respectively the set $H \backslash G$ of its left cosets); therefore the $\mathscr{D}$-class of H is a square of side $(G : N_G(H))$ of $\mathscr{H}$-classes;
- the maximal subgroup of $\mathscr{P}'(G)$ containing the element H is
$$(G/H) \cap (H\backslash G) = \{Hx : x \in N_G(H)\}$$
and is therefore isomorphic with the quotient $N_G(H)/H$;
- the $\mathscr{D}$-class of H may be represented as

$H\ *$ $Hx{=}xH$	$\cdots$	Hy Hz	$\cdots$
$\vdots$	$\ddots$	$\vdots$	
$y^{-1}H$ $z^{-1}H$	$\cdots$	$y^{-1}Hy\ *$ $y^{-1}Hz$	$\cdots$
$\vdots$		$\vdots$	$\ddots$

  where a $*$ marks an idempotent; in particular, the blocks of $\mathscr{P}'(G)$ are groups;
- $B_2 \prec \mathscr{P}'(G)$ if and only if $\mathscr{P}'(G)$ has a regular $\mathscr{D}$-class which is not a group, i.e., G has a subgroup which is not normal;
- if G is a non-Abelian group, then $G \times G$ has some subgroup which is not normal.

We thus have the following result.

Proposition 2.2. *For every finite non-Abelian group G, B_2 divides $\mathscr{P}'(G \times G)$.*$\square$

2.3. Completely simple semigroups

A *rectangular band* is a direct product of a left-zero semigroup by a right-zero semigroup. A *rectangular group* is a regular semigroup whose idempotents form a rectangular band, i.e., a direct product of a rectangular band by a group.

It is easily checked that, if $S = L \times R$, where L is a left-zero semigroup and R is a right-zero semigroup, then the semigroup $\mathscr{P}'(S)$ is not regular but its regular elements form the rectangular band $\mathscr{P}'(L) \times \mathscr{P}'(R)$. More generally, in the finite case, we have the following result.

Proposition 2.3. *If $S = L \times G \times R$ be a finite rectangular group, where and L and R are as above and G is a group, then the blocks of $\mathscr{P}'(S)$ are (isomorphic with) rectangular groups of the form $\mathscr{P}'(L) \times N_G(H)/H \times \mathscr{P}'(R)$ where H is a subgroup of G.*

Proof. Suppose A is an idempotent of $\mathscr{P}'(S)$. Then, since A is a subsemigroup of S, A contains some idempotent $(e,1,f)$. Let $\{e\} \times H \times \{f\}$ be the maximal subgroup of A containing $(e,1,f)$. Since each nonempty intersection $A \cap \{e'\} \times G \times \{f'\}$ ($e' \in L$, $f' \in R$) is a subgroup of A, A is a union of subgroups. Moreover, $A \cap \{e'\} \times G \times \{f'\} \supseteq \{e'\} \times H \times \{f'\}$ and so $A = L' \times H \times R'$ for some $L' \subseteq L$ and $R' \subseteq R$.

The $\mathscr{R}$-class of $L' \times H \times R'$ consists of all subsets of S of the form $L' \times Hy \times R''$ with $y \in G$ and $\emptyset \neq R'' \subseteq R$. The $\mathscr{H}$-class of such an element of $\mathscr{P}'(S)$ is a subgroup if and only if $y \in N_G(H)$.

Hence each block of $\mathscr{P}'(S)$ is isomorphic with a rectangular group of the form $\mathscr{P}'(L) \times N_G(H)/H \times \mathscr{P}'(R)$.□

For the general completely simple case, the situation is more complicated. Even if the subgroups are Abelian, the blocks need not be completely simple. Take for instance the Rees matrix semigroup

$$S = \mathscr{M}(\{1,2\}, G, \{1,2\}, \begin{pmatrix} 1 & 1 \\ 1 & b \end{pmatrix})$$

where $G = \langle a, b \rangle$ is a Klein four-group. Consider the following elements of $\mathscr{P}'(S)$:

$$\begin{aligned} T &= \{(1,1,1),(1,a,1)\} \cup \{(1,1,2),(1,a,2)\} \\ U &= \{(2,1,1),(2,a,1)\} \\ V &= \{(2,1,1),(2,a,1)\} \cup \{(2,1,2),(2,a,2)\}. \end{aligned}$$

Then T and U are idempotents, $UT = V$, $\{(1,1,1)\}V = T$, and $V\{(1,1,1)\} = U$ so that the block B containing T also contains U by Green's lemma. Now, we have

$$TU = \{(1,1,1),(1,a,1),(1,b,1),(1,ab,1)\}.$$

Hence every element of $\mathscr{P}'(S)$ which lies $\mathscr{J}$-below TU must have either 0 or (at least) 4 elements in each $\mathscr{H}$-class. This shows that B is not simple.

3. Pseudovarieties and various operators

The examples in section 2 suggest that we consider classes of finite semigroups closed under taking divisors and finitary direct products. Such classes are called *pseudovarieties*. Throughout this paper, we use bold capital letters to denote pseudovarieties. In particular, the letters **V**, **W**, **X** are generally used to denote arbitrary pseudovarieties.

For a class $\mathscr{C}$ of finite semigroups, denote respectively by $\mathcal{H}\mathscr{C}$, $\mathcal{S}\mathscr{C}$, $\mathcal{P}_{\text{fin}}\mathscr{C}$, and $\mathcal{V}\mathscr{C}$ the classes consisting of all homomorphic images, all subsemigroups, all finitary direct products of members of $\mathscr{C}$, and the pseudovariety generated by $\mathscr{C}$. So, in particular, $\mathcal{V} = \mathcal{H}\mathcal{S}\mathcal{P}_{\text{fin}}$.

Pseudovarieties admit several sorts of equational descriptions but more complicated in general than the classical Birkhoff theorem for varieties of algebras. The most useful such description seems to be the one given by Reiterman [52]: every pseudovariety is defined by a set of "pseudoidentities". Since there are several basic presentations of Reiterman's result (e.g., [52, 11, 18, 34]), rather than defining pseudoidentities, let us just say that pseudoidentities are formal equalities between certain types of expressions generalizing semigroup terms (i.e., words). The new expressions which are relevant in most of this paper are those which are built from words using the ω-*power*. The ω-power u^ω is to be (recursively) interpreted in a finite semigroup as giving the only idempotent power of the interpretation of u. For a nonnegative integer k, we abbreviate $u^\omega u^k$ by $u^{\omega+k}$.

For a set Σ of pseudoidentities, we denote by $[\![\Sigma]\!]$ the class of all finite semigroups which verify all pseudoidentities from Σ; this class is a pseudovariety and Σ is said to be a *basis of pseudoidentities* for it. The letter variables appearing in a pseudoidentity will usually be chosen to be $x, y, z, t, \ldots$. To make pseudoidentities more compact and hopefully easier to read, we convention that when we write $e, f, g, \ldots$ in a pseudoidentity, these letters stand for expressions of the form $x_e^\omega, x_f^\omega, x_g^\omega, \ldots$ where $x_e, x_f, x_g, \ldots$ are new variables that previously did not intervene in the pseudoidentity. So, for instance, $[\![xe = ex]\!]$ is the pseudovariety consisting of all finite semigroups whose idempotents are central. Another two convenient conventions are to write $u = 1$ and $u = 0$ respectively for $uy = yu = y$ and $uy = yu = u$ where y is variable that does not appear in the expression u.

By an *identity* we mean a pseudoidentity whose sides are both words. The set of letters occurring in a word u is denoted by $c(u)$. An identity $u = v$ is *linear* if u and v are both products of distinct variables; if further $c(u) = c(v)$, then $u = v$ is said to be a *permutation identity*. We say that a pseudovariety $\mathbf{V}$ is *linear* if it is a union of pseudovarieties of the form $[\![\Sigma]\!]$ where Σ is a set of linear identities.

Denote by $\mathbf{S}$ the pseudovariety of all finite semigroups. For a semigroup S, let $E(S)$ be the set of its idempotents; for $X \subseteq S$, let $\langle X\rangle$ denote the subsemigroup generated by X.

For semigroups S and T, we denote by $S * T$ a semidirect product of S by T which is determined by a monoid homomorphism from T^1 into the monoid of endomorphisms of S (cf. [11]).

Before introducing further examples of pseudovarieties, we proceed to define some important operators on pseudovarieties. As argued in [11], most of the contemporary theory of finite semigroups can be viewed as being centered essentially on the study of such operators. For pseudovarieties $\mathbf{V}$ and $\mathbf{W}$, let

$$\begin{aligned}
\mathcal{B}\mathbf{V} &= \{S \in \mathbf{S} : \text{every block of } S \text{ lies in } \mathbf{V}\}\\
\mathcal{D}\mathbf{V} &= \{S \in \mathbf{S} : \text{every regular } \mathscr{D}\text{-class of } S \text{ is a subsemigroup which lies in } \mathbf{V}\}\\
\mathcal{E}\mathbf{V} &= \{S \in \mathbf{S} : \langle E(S)\rangle \in \mathbf{V}\}\\
\mathcal{L}\mathbf{V} &= \{S \in \mathbf{S} : eSe \in \mathbf{V} \text{ for all } e \in E(S)\}\\
\mathcal{L}in\mathbf{V} &= \bigcap\{\mathbf{U} : \mathbf{V} \subseteq \mathbf{U},\ \mathbf{U} \text{ is a linear pseudovariety}\}\\
\mathcal{M}\mathbf{V} &= \mathcal{HSP}_{\text{fin}}\{S^1 : S \in \mathbf{V}\}\\
\mathcal{P}\mathbf{V} &= \mathcal{HSP}_{\text{fin}}\{\mathscr{P}(S) : S \in \mathbf{V}\}\\
\mathcal{P}'\mathbf{V} &= \mathcal{HSP}_{\text{fin}}\{\mathscr{P}'(S) : S \in \mathbf{V}\}\\
\mathcal{P}d\mathbf{V} &= \{S \in \mathbf{S} : \text{all cyclic subgroups of } S \text{ lie in } \mathbf{V}\}\\
\mathbf{V} \vee \mathbf{W} &= \mathcal{HSP}_{\text{fin}}(\mathbf{V} \cup \mathbf{W}) = \mathcal{HS}\{S \times T : S \in \mathbf{V}, T \in \mathbf{W}\}\\
\mathbf{V} * \mathbf{W} &= \mathcal{HSP}_{\text{fin}}\{S * T : S \in \mathbf{V}, T \in \mathbf{W}\} = \mathcal{HS}\{S * T : S \in \mathbf{V}, T \in \mathbf{W}\}
\end{aligned}$$

$$\mathbf{V} \circledcirc\!\!\!\!\!\;m\; \mathbf{W} = \mathcal{HS}\{S \in \mathbf{S} : (\exists T \in \mathbf{W})\,(\exists \text{ a homomorphism } \varphi : S \to T)$$
$$(\forall e \in E(T))\, e\varphi^{-1} \in \mathbf{W}\}.$$

Here are now a number of examples of pseudovarieties and their definitions by pseudoidentities. Some of the results summarized below are not trivial.

$$\begin{aligned}
\mathbf{A} &= \{\text{finite aperiodic semigroups}\} = [\![x^{\omega+1} = x\omega]\!]\\
\mathbf{Ab} &= \{\text{finite Abelian groups}\} = [\![e = 1,\, xy = yx]\!]\\
\mathbf{B} &= \{\text{finite bands}\} = [\![x^2 = x]\!]\\
\mathcal{B}\mathbf{G} &= [\![(ef)^\omega = (fe)^\omega]\!]\\
\mathbf{Com} &= \{\text{finite commutative semigroups}\} = [\![xy = yx]\!]\\
\mathbf{CR} &= \{\text{finite completely regular semigroups}\} = [\![x^{\omega+1} = x]\!]\\
\mathbf{CS} &= \{\text{finite (completely) simple semigroups}\} = [\![x^{\omega+1} = x,\, (xy)^\omega x = x]\!]\\
\mathbf{D} &= [\![xe = e]\!]\\
\mathbf{D}_1 &= \{\text{finite right-zero semigroups}\} = [\![xy = y]\!]\\
\mathcal{D}\mathbf{A} &= [\![(xy)^\omega (yx)^\omega (xy)^\omega = (xy)^\omega,\, x^{\omega+1} = x^\omega]\!]\\
\mathcal{D}\mathbf{S} &= [\![((xy)^\omega (yx)^\omega (xy)^\omega)^\omega = (xy)^\omega]\!]\\
\mathcal{ED}\mathbf{S} &= \mathcal{B}\mathbf{CS} = [\![(e(fxe)^\omega f)^{\omega+1} = e(fxe)^\omega f]\!] \text{ [11]}\\
\mathbf{G} &= \{\text{finite groups}\} = [\![e = 1]\!]\\
\mathbf{I} &= \{\text{trivial semigroups}\} = [\![x = y]\!]\\
\mathbf{J} &= \{\text{finite } \mathscr{J}\text{-trivial semigroups}\} = [\![(xy)^\omega x = (xy)^\omega = y(xy)^\omega]\!]\\
\mathbf{K} &= [\![ex = e]\!]\\
\mathbf{K}_1 &= \{\text{finite left-zero semigroups}\} = [\![xy = x]\!] = \mathbf{B} \cap \mathbf{K}\\
\mathcal{L}\mathbf{I} &= \{\text{finite locally trivial semigroups}\} = [\![exe = e]\!]\\
\mathcal{M}\mathbf{D} &= [\![exe = xe]\!] \text{ [45]}\\
\mathcal{M}\mathbf{K} &= [\![exe = ex]\!] \text{ [45]}\\
\mathcal{M}\mathbf{K}_1 &= [\![x^2 = x,\, xyx = xy]\!] = \mathbf{B} \cap \mathbf{R}\\
\mathcal{ML}\mathbf{I} &= [\![x^\omega yxzx^\omega = x^\omega yzx^\omega,\, x^\omega yxztz^\omega = x^\omega yzxtz^\omega]\!] \text{ [4]}\\
\mathcal{M}\mathbf{N} &= [\![ex = xe,\, x^{\omega+1} = x^\omega]\!] = \mathbf{ZE} \cap \mathbf{A} \text{ [59]}\\
\mathcal{M}\mathbf{NB} &= [\![x^2 = x,\, xyxzx = xyzx]\!] = \mathbf{B} \cap \mathcal{ML}\mathbf{I}\\
\mathbf{N} &= \{\text{finite nilpotent semigroups}\} = [\![e = 0]\!]\\
\mathbf{NB} &= \{\text{finite normal bands}\} = [\![x^2 = x, xyzx = xzyx]\!] = \mathbf{B} \cap \mathbf{Perm}\\
\mathbf{Perm} &= \{\text{finite permutative semigroups}\} = [\![exyf = eyxf]\!]\\
\mathbf{R} &= \{\text{finite } \mathscr{R}\text{-trivial semigroups}\} = [\![(xy)^\omega = (xy)^\omega]\!]\\
\mathbf{ReG} &= \{\text{finite rectangular groups}\} = [\![x^{\omega+1} = x,\, efe = e]\!]\\
\mathbf{RO} &= \{\text{finite regular orthogroups}\} = [\![x^{\omega+1} = x,\, xyx^\omega zx = xyzx]\!] \text{ [44]}\\
\mathbf{Sl} &= \{\text{finite semilattices}\} = [\![x^2 = x,\, xy = yx]\!]\\
\mathcal{V}\{B_2\} &= \mathcal{V}\{B_n\} = [\![x^3 = x^2,\, x^2y^2 = y^2x^2,\, xyxyx = xyx]\!] \text{ [65, 66]}\\
\mathbf{ZE} &= [\![xe = ex]\!] = \mathbf{G} \vee \mathbf{Com} \text{ [5]}.
\end{aligned}$$

4. Problems

In applications of finite semigroup theory, one is interested in obtaining algorithms to determine whether a finite semigroup belongs to a certain pseudovariety $\mathbf{V}$. More generally, we say that a class $\mathscr{C}$ of finite semigroups is *decidable* if there is an algorithm to decide whether a finite semigroup belongs to it.

We now list a number of questions involving operators on pseudovarieties.

Decidability. Let us call *decidability question* for an operator $\mathcal{O}$ the question as to whether, for every decidable pseudovariety $\mathbf{V}$, $\mathcal{O}\mathbf{V}$ is also decidable. The basic difficulty of finite semigroup theory is that most often, the operators on pseudovarieties are defined in terms of generators and the decidability question for the operator $\mathcal{V}$ has a negative answer. This is known to happen even under further restrictions, namely for the operators $\vee$, $*$, and ⓜ the decidability question also has a negative answer [1, 53].

For other operators, such as $\mathcal{B}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{L}$, and $\mathcal{P}d$, it is easy to show that the decidability question has an affirmative answer. Often, calculations involving these operators end up giving casuistic relationships between them.

Image. Another type of problem which is of interest consists in determining the image of a given operator. For the operators $\mathcal{B}$, $\mathcal{D}$, and $\mathcal{E}$, the problem of determining the image does not seem to have been considered. For instance, given a pseudovariety $\mathbf{V}$ (say by a defining set of pseudoidentities or by an algorithm to test membership in $\mathbf{V}$), determine if it is of the form $\mathbf{V} = \mathcal{E}\mathbf{W}$ for some pseudovariety $\mathbf{W}$.

Equations. A generalization of the image problem consists in solving equations involving the operators. For example, the equation $\mathbf{X} * \mathbf{D} = \mathcal{L}\mathbf{X}$ has attracted considerable attention and it has played an important role in understanding semidirect products [28, 60, 64, 11]. It remains an open question what are the idempotent pseudovarieties with respect to the operation $*$ (which is associative). Disproving conjectures proposed in [11], L. Teixeira [62] has recently showed that there are uncountably many such idempotents among the aperiodic pseudovarieties.

Irreducibility. For the binary operators $\vee$, $*$, and ⓜ, there is a related question which consists in determining which pseudovarieties decompose nontrivially using these operators. The paper [38] contains a class of examples of indecomposable pseudovarieties. See also [11] for further information on the operators $\vee$ and $*$.

Basis. Another type of problem that has received considerable attention is to determine a basis of pseudoidentities for the result of applying an operator to pseudovarieties for which bases are known. Formulated as such, this does not appear to be relevant to obtain decidability applications. However, the main result in [19] is of this type and has led to substantial decidability applications.

Finite basis. Assuming that the operands have finite bases of pseudoidentities, we ask whether applying one of the operators the resulting pseudovariety also has a finite basis. For most operators the answer is negative.

Computable finite basis. This is similar to the preceding problem but it is assumed that the bases for the operands only involve expressions which can be effectively computed, and this further requirement is also sought for the result of applying the operators.

Preservation of $\vee/\cap$. An affirmative answer for this kind of question can be extremely useful, shedding new light on the lattice of pseudovarieties [22]. For the operators considered here, an affirmative answer in both cases does not seem very likely.

Number of iterates. For a unary operator $\mathcal{O}$, we ask how many different operators can be obtained by repeatedly composing $\mathcal{O}$ with itself. This measures on one hand how far the operator is from being an idempotent, and on the other hand, the potential for the operator to yield a nontrivial decomposition theory. From this point of view, the unary operators $\mathscr{P}$ and $\mathscr{P}'$ are particularly interesting.

Although this paper is devoted mostly to the power operators, since there are so many relationships among the operators of section 3, it is worthwhile considering these questions for them. Some of the questions are somewhat meaningless or trivial for some of the operators. We gather a summary of results in Table 1 where

- a blank entry means that the author is unware of any material in the literature which is directly relevant;
- a question mark in an entry means that the problem has, at least implicitly, been considered in the literature;
- a Y/N entry means, respectively, that the answer for the question in the line containing it for the operator in the column containing it is affirmative/negative;
- references in the heading for a row or column apply to the whole of it;
- in general, the indicated references either directly justify the entry or contain related material;
- in the presence of a Y/N and the absence of references, the answer is elementary;
- in the "Image" row, $(\mathbf{V})$ in an entry means that there is a natural bijection between the image of the operator in the corresponding column and the lattice of subpseudovarieties of $\mathbf{V}$.

5. A review of partial results

We now concentrate on the power operators and review the main results concerning them. Earlier surveys on the power operator can be found in [47, 7, 11].

Since we are concerned here only with the semigroup theoretic aspects of the operator, we will not go into details on the relationships with some natural operators on classes of rational languages which actually provided the initial motivation for the study of the power operators. We refer the reader to [11, Section 11.2] for an introduction to this facet and for further references.

	$\mathcal{B}$	$\mathcal{D}$	$\mathcal{E}$	$\mathcal{L}$	$\mathcal{L}in$ [11]	$\mathcal{M}$ [11]	$\mathcal{P}$ [11]	$\mathcal{P}'$ [11]	$\mathcal{P}d$ [11]
Decidability	Y	Y	Y	Y		?	?	?	Y
Basis		Y	Y	Y	Y	Y	?	?	Y
Finite basis		Y	N [68, 69]	Y	Y	Y	N	N	Y
Computable finite basis		Y	–	Y	?	?	–	–	Y
Image		(**CS**)					?	?	(**Ab**)
Preserves $\vee$		N [14, 17]		? [11]	N	Y	N	N	
Preserves $\cap$	Y	Y	Y	Y	N	N	N	N	Y
Number of iterates	1	1	1	1	1	1	3	3	1

	$\vee$	$*$	ⓜ
Decidability [1, 53, 56]	N [12]	N [12]	N [50]
Basis	N [11, 58]	N [19, 12, 16]	N [50, 56]
Finite basis [68, 67]	N	N	N
Preserves $\vee$ (fixing left variable)	Y		
Preserves $\vee$ (fixing right variable)	Y	Y [11]	
Preserves $\cap$ (fixing left variable)	N	N	
Preserves $\cap$ (fixing right variable)	N	[19, 15]	Y [50]

Table 1: A summary of properties and problems concerning various operators on pseudovarieties

Another kind of motivation that has prompted developments in the area is the challenge to deal with an apparently computationally intractable operator. Due to its exponential character, it is inviable to work out complete examples before guessing what is going on, since very small examples have limited importance.

The first easy remark consists in noting that dealing with the empty set instead of just with nonempty subsets of a semigroup just means adding an extra zero, which amounts, up to division, to taking a product with a semilattice. Moreover, it is easy to check that $\mathcal{P}'\mathcal{L}\mathbf{I} \subseteq \mathcal{L}\mathbf{I}$ and, if S contains a nontrivial monoid, then $\mathscr{P}'(S)$ contains a nontrivial semilattice. This allows us to relate the operators $\mathcal{P}'$ and $\mathcal{P}$ as follows.

Lemma 5.1. *a)* $\mathcal{P}\mathbf{V} = \mathcal{P}'\mathbf{V} \vee \mathbf{Sl}$.
b) $\mathcal{P}'\mathbf{V} = \mathcal{P}\mathbf{V} \Leftrightarrow \mathbf{V} \not\subseteq \mathcal{L}\mathbf{I}$.

In view of Lemma 5.1(b), we will only refer to both operators $\mathcal{P}'$ and $\mathcal{P}$ when dealing with pseudovarieties of locally trivial semigroups.

The next observation, which seems to be due to S. Margolis, follows from Proposition 2.1.

Proposition 5.2. $\mathcal{P}'\mathcal{V}\{B_2\} = \mathbf{S}$.$\square$

Conversely, we have the following solution of an equation on pseudovarieties for which a proof using pseudoidentities can be found in [9]. The original result concerned pseudovarieties of monoids, instead of semigroups. But, as observed in [11, Exercise 11.10.1], this fact really makes no difference.

Theorem 5.3 [39]. *A pseudovariety* $\mathbf{X}$ *satisfies the equation* $\mathcal{P}\mathbf{X} = \mathbf{S}$ *if and only if* $B_2 \in \mathbf{X}$.

From Proposition 2.2, we in turn obtain the following formulation in terms of the power operators.

Proposition 5.4. *If* $\mathbf{V}$ *contains some non-Abelian group, then* $\mathcal{P}^2\mathbf{V} = \mathbf{S}$.$\square$

More generally, the presence of a non-commutative monoid accounts for a somewhat slower blowup in the power.

Theorem 5.5 [41]. *If* $\mathbf{V}$ *contains a non-commutative monoid, then* $\mathcal{P}^3\mathbf{V} = \mathbf{S}$.

Since $\mathcal{P}\mathbf{Com} \subseteq \mathbf{Com}$, for pseudovarieties of monoids Theorem 5.5 completes the picture of possible blowups, the converse being also true. For semigroups, the situation is more complicated.

Theorem 5.6 [3]. *If* $\mathbf{V}$ *contains some non-permutative semigroup, then* $\mathcal{P}^3\mathbf{V} = \mathbf{S}$.

The converse again follows from the simple observation that $\mathcal{P}\mathbf{Perm} \subseteq \mathbf{Perm}$. But it was only a few years later that a more complete picture became available concerning the behaviour of the power operators on pseudovarieties of permutative semigroups.

Theorem 5.7 [11]. *The value of the power operator on pseudovarieties of permutative semigroups is given by the formula*

$$\mathcal{P}'\mathbf{V} = \mathcal{L}in\mathbf{V} \cap \mathcal{P}d\mathbf{V} = \mathcal{P}'^2\mathbf{V}. \tag{1}$$

The extent to which the above formula is effective depends on the ability to compute the operators $\mathcal{L}in$ and $\mathcal{P}d$. As to the operator $\mathcal{P}$, if $\mathbf{V} \subseteq \mathcal{L}\mathbf{I}$, then $\mathcal{P}d\mathbf{V} = \mathbf{A}$ and so $\mathcal{P}'$ coincides with $\mathcal{L}in$ on pseudovarieties of locally trivial semigroups. The join of linear pseudovarieties with $\mathbf{Sl}$ is not hard to compute [11, Proposition 6.3.16], thus reducing the calculation of $\mathcal{P}$ to that of $\mathcal{P}'$.

As a corollary of Theorems 5.6 and 5.7, one may deduce the following.

Theorem 5.8 [3]. *The semigroup of operators on pseudovarieties generated by $\{\mathcal{P}', \mathcal{P}\}$ has eight elements and is defined by the following relations:*

$$\mathcal{P}'\mathcal{P} = \mathcal{P}^2,\ \mathcal{P}^2\mathcal{P}' = \mathcal{P}^3,\ \mathcal{P}'^4 = \mathcal{P}'^3,\ \mathcal{P}'^3\mathcal{P} = \mathcal{P}'^2\mathcal{P},\ \mathcal{P}\mathcal{P}'^3 = \mathcal{P}\mathcal{P}'^2.$$

As to the groups present in power semigroups, the following result says that no new groups show up if we avoid B_2. Note that a pseudovariety does not contain B_2 if and only if it is contained in $\mathcal{D}\mathbf{S}$. For a pseudovariety $\mathbf{H}$ of groups, let $\overline{\mathbf{H}}$ denote the pseudovariety consisting of all finite semigroups whose subgroups lie in $\mathbf{H}$.

Theorem 5.9 [39]. *If $\mathbf{H}$ is a pseudovariety of groups, then $\mathcal{P}\mathbf{V} \subseteq \mathcal{P}\mathcal{D}\mathbf{S} \cap \overline{\mathbf{H}}$ if and only if $\mathbf{V} \subseteq \mathcal{D}\mathbf{S} \cap \overline{\mathbf{H}}$.*

In particular, the operator $\mathcal{P}$ restricted to pseudovarieties of groups is injective. The apriodic case of Theorem 5.9 is worth mentioning separately.

Corollary 5.10 [51]. $\mathcal{P}\mathbf{V} \subseteq \mathbf{A} \Leftrightarrow \mathbf{V} \subseteq \mathcal{D}\mathbf{A}$.

We showed in subsection 2.2 that $\mathcal{P}\mathbf{G} \subseteq \mathcal{B}\mathbf{G}$. Margolis and Pin [42] proved the following chain of inclusions and equalities:

$$\mathcal{P}\mathbf{G} = \mathbf{J} * \mathbf{G} \subseteq \mathbf{J} \textcircled{m} \mathbf{G} = \mathcal{E}\mathbf{J} = \mathcal{B}\mathbf{G}$$

and asked whether they are all equalities. This was confirmed by Henckell and Rhodes [33] (see also [32, 48]) using results of Knast [37] and Ash [21], both of which are considered to be very deep and hard results.

Theorem 5.11 [33]. $\mathcal{P}\mathbf{G} = \mathcal{B}\mathbf{G}$.

A related equation has also been solved.

Theorem 5.12 [11]. $\mathcal{P}\mathbf{X} = \mathcal{B}\mathbf{G} \Leftrightarrow \mathbf{G} \subseteq \mathbf{V} \subseteq \mathbf{ZE}$.

For a more refined analysis of the behaviour of the power operators on pseudovarieties containing non-permutative semigroups, following the earlier work of Margolis and Pin [41, 45], the author endeavored to compute the powers of the minimal non-permutative pseudovarieties which he had obtained [3] extending the monoid case [41]. These pseudovarieties are generated either by a finite non-Abelian group (and the necessary groups have been determined in [2, 11]), one of the semigroups presented by

$$\begin{aligned}
N^1 &= \langle a,b; a^2=b^2=ba=0\rangle^1 \\
B(1,2)^1 &= \langle e,f; ef=ee=e,\ fe=ff=f\rangle^1 \\
B(2,1)^1 &= \langle e,f; ef=ff=f,\ fe=ee=e\rangle^1 \\
Y &= \langle e,s,f; e^2=e,\ f^2=f,\ es=sf=s,\ ef=fe=0\rangle \\
Q &= \langle e,s,t; e^2=e,\ es=s,\ te=t,\ se=et=ts=0\rangle,
\end{aligned}$$

or one of the following Rees matrix semigroups, where p denotes a prime integer and Z_p a cyclic group of order p generated by a:

$$K_p = \mathscr{M}(2, Z_p, 2, \begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix}).$$

In the group case, perhaps it is more enlightening to give the minimal pseudovarieties containing a non-Abelian group in terms of pseudoidentities. To do so, for $k \geq 1$, denote by $s^{\omega-k}$ the power of an element s of a finite semigroup which is the inverse of $s^{\omega+k}$ in the cyclic subgroup generated by this element, and define length 2 and 3 commutators by $[x,y] = x^{\omega-1}y^{\omega-1}xy$ and $[x,y,z] = [[x,y],z]$. The pseudovarieties in question are then the following [11]:

$$\begin{aligned}
&[\![x^4=1,\ [x,y]^2=q,\ [x,y,z]=1]\!] \\
&[\![x^p=1,\ [x,y,z]=1]\!] \quad \text{for a prime } p>2 \\
&[\![x^{pq}=1,\ [x^q,y^q]=1,\ [x,y]^p=1]\!] \quad \text{for } p \text{ and } q \text{ distinct primes.}
\end{aligned}$$

There seems to have been no progress in the group case beyond Theorem 5.11 and the Abelian case (which is included in Theorem 5.7).

For $\mathcal{V}(N^1)$, we have the following result.

Theorem 5.13 [45]. *a)* $\mathcal{P}\mathbf{V} \subseteq \mathbf{J} \Leftrightarrow \mathbf{V} \subseteq \mathcal{M}\mathbf{N}$.
b) $\mathcal{P}\mathbf{V} = \mathbf{J} \Leftrightarrow N^1 \in \mathbf{V}$ *and* $\mathbf{V} \subseteq \mathcal{M}\mathbf{N}$.
c) $\mathcal{P}\mathbf{V} \subseteq \mathbf{R} \Leftrightarrow \mathbf{V} \subseteq \mathcal{M}\mathbf{K}$.

The powers of $\mathcal{V}(B(1,2)^1) = \mathcal{M}\mathbf{K}_1$ and $\mathcal{M}\mathbf{K}$ (and so their duals) have also been computed [11], along with all powers of pseudovarieties contained in $\mathcal{M}\mathbf{K}$ but not in **Perm**. The description of such calculations would be too long and technical to be included here. It actually contains some errors which will be hopefully corrected in a forthcoming paper. We will just mention here that both $\mathcal{P}\mathcal{M}\mathbf{K}_1$ and $\mathcal{P}\mathcal{M}\mathbf{K}$ are decidable pseudovarieties and there is no solution to the equation $\mathcal{P}\mathbf{X} = \mathbf{R}$.

A related result which is relevant in view of Corollary 5.10 is the following.

Theorem 5.14 [13]. *For a pseudovariety* $\mathbf{V}$, *we have* $\mathcal{P}\mathbf{V} \subseteq \mathcal{D}\mathbf{A}$ *if and only if* $\mathbf{V} \subseteq \mathcal{M}\mathbf{K} \vee \mathbf{D}$, $\mathbf{V} \subseteq \mathbf{K} \vee \mathcal{M}\mathbf{D}$, *or* $\mathbf{V} \subseteq [\![exeyxfyf = exyf]\!]$.

Confirming the author's guess put forward in [11], the following computation (for which of course there is a dual result for $\mathbf{K} \vee \mathcal{M}\mathbf{D}$) was achieved by Azevedo and Zeitoun [23]:

$$\mathcal{M}\mathbf{K} \vee \mathbf{D} = [\![exeyf = exyf,\, x^{\omega+1} = x^{\omega}]\!].$$

Following [46], let us call the *exponent* of a pseudovariety not contained in **Perm** the least nonnegative integer n such that $\mathcal{P}^n\mathbf{V} = \mathbf{S}$, where we let $\mathcal{P}^0\mathbf{V} = \mathbf{V}$. By Theorem 5.6, the exponent of a pseudovariety is either undefined or at most 3. The following result extends the calculation of the exponent for pseudovarieties of aperiodic monoids which follows from the results of [10]. To present it, we need to introduce some more small semigroups. Denote by $F_n\mathbf{V}$ the free semigroup on n generators in the variety generated by $\mathbf{V}$. Consider also the semigroup with the following presentation:

$$I = \langle e, s, t; e^2 = e,\, es = s,\, s^2 = se,\, et = st,\, te = ts = t^2 = t\rangle.$$

For a semigroup S, let S^ρ denote its *dual* which is obtained by transposing the table of multiplication of S.

Theorem 5.15 [13]. *For a non-permutative pseudovariety* $\mathbf{V}$ *of aperiodic semigroups,*

a) $\mathbf{V}$ *does not have exponent 0;*

b) $\mathbf{V}$ *has exponent 1 if and only if* $B_2 \in \mathbf{V}$*;*

c) $\mathbf{V}$ *has exponent 2 if and only if* $B_2 \notin \mathbf{V}$ *and* $\mathbf{V}$ *contains at least one of the semigroups* Y, Q, I, I^ρ, $F_3[\![x^2 = x, xyzxz = xyz]\!]$, *and* $F_3[\![x^2 = x, xzxyz = xyz]\!]$*;*

d) $\mathbf{V}$ *has exponent 3 if and only if* $\mathbf{V}$ *contains none of the semigroups in (c).*

An interesting connection with language theory is given by the following result which is essentially an application of Schützenberger's characterization of the languages recognized by semigroups in $\mathcal{D}\mathbf{A}$ [54].

Proposition 5.16 [49]. *The languages recognized by semigroups in* $\mathcal{P}\mathcal{D}\mathbf{A}$ *are the rational languages of dot-depth 2.*

It remains an open problem to decide when a rational language has dot-depth 2 [61], i.e., whether $\mathcal{P}\mathcal{D}\mathbf{A}$ is decidable. As far as the approach to this problem by using power semigroups is concerned, the following result seems relevant.

Theorem 5.17 [6]. $\mathcal{P}\mathbf{X} = \mathcal{P}\mathcal{D}\mathbf{A} \Leftrightarrow Y \in \mathbf{V}$ *and* $\mathbf{V} \subseteq \mathcal{D}\mathbf{A}$.

The semigroup Y also plays a role in the following result.

Theorem 5.18 [8]. *The following conditions on a pseudovariety* $\mathbf{V}$ *are equivalent:*

i) $\mathcal{P}\mathbf{V} \subseteq \mathbf{Com} * \mathbf{D}$*;*

ii) $\mathcal{P}\mathbf{V} \subseteq \mathcal{L}\mathbf{Com}$*;*

iii) $\mathbf{V} \subseteq \mathcal{L}\mathbf{Com}$ *and* $Y \notin \mathbf{V}$*;*

iv) $\mathbf{V} \subseteq \mathcal{L}\mathbf{Com} \cap [\![(ef)^{\omega}exf = exf]\!]$.

In connection with Theorem 5.18, it is worth mentioning the following result of Thérien and Weiss [63]:

$$\mathbf{Com} * \mathbf{D} = [\![exfyezf = ezfyexf]\!] \subsetneqq \mathcal{L}\mathbf{Com}.$$

The aperiodic case is also of interest and yields the solution of another equation involving one of the generators of minimal non-permutative pseudovarieties.

Theorem 5.19 [8]. *$\mathcal{P}\mathbf{X} = (\mathbf{Com}*\mathbf{D}) \cap \mathbf{A} \Leftrightarrow Q \in \mathbf{V}$ and $\mathbf{V} \subseteq \mathcal{L}\mathbf{Com} \cap [\![efexf = exf]\!] \cap \mathbf{A}$.*

For pseudovarieties of monoids, there are further simplifications which have already allowed the author to give a complete picture of the image of the power operator on pseudovarieties of aperiodic monoids [10] which we now describe. The result, which turns out to be a modular lattice under inclusion, departs from the description of the lattice of pseudovarieties of band monoids which was obtained by Wismath [70]. Both lattices are depicted in Figure 1, where we denote by $\mathbf{M}$ the pseudovariety of all finite monoids.

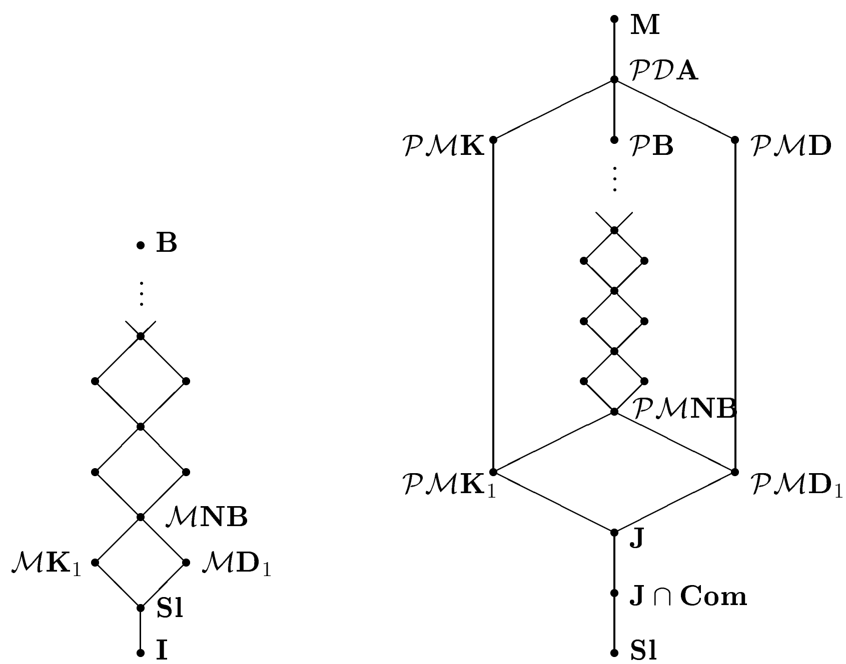

Figure 1: The lattice of pseudovarieties of band monoids and
the lattice of powers (under $\mathcal{P}'$) of pseudovarieties of aperiodic monoids

In particular, the power operator is injective when restricted to pseudovarieties of band monoids. In the picture of the lattice of powers, all pseudovarieties in the interval between

$\mathcal{PM}\mathbf{NB}$ and $\mathcal{PD}\mathbf{A}$ remain unknown. The corresponding picture for the semigroup case is still unknown. The extra complications are due to at least two reasons. First, the interval between $\mathcal{PM}\mathbf{K}_1$ and $\mathcal{PM}\mathbf{K}$ becomes nontrivial [11]. Second, the lattice of pseudovarieties of bands is more complicated than its monoid counterpart [70].

The above results seem to indicate that, apart from the pseudovariety $\mathcal{PD}\mathbf{A}$, most of the interesting questions on the power operator will involve the calculation of the powers of pseudovarieties of completely regular semigroups. At present, not much is known in this direction.

For the pseudovariety $\mathcal{P}\mathbf{CR}$ itself, the following upper bound is given in [11, Exercise 11.11.7]:

$$(x^{\omega+1}y^{\omega+1})^{\omega} = [x^{\omega+1}(yx)^{\omega}y^{\omega+1}]^{\omega}.$$

But this is a rather ad hoc upper bound which nothing leads to believe might be strict.

For $\mathcal{P}(\mathbf{CS}\cap\overline{\mathbf{Ab}})$, a little refinement of the arguments in [11, Section 11.9], which we now present, gives an interesting upper bound. First of all, it is an elementary exercise to show that $\mathbf{CS}\cap\overline{\mathbf{Ab}} = \mathbf{Ab} * \mathbf{D}_1$. By Theorem 5.18, it follows that

$$\mathcal{P}(\mathbf{CS}\cap\overline{\mathbf{Ab}}) \subseteq \mathbf{Com} * \mathbf{D}.$$

Using [11, Corollary 10.7.4] or a direct calculation, it is then an easy matter to show that $\mathbf{CS}\cap\overline{\mathbf{Ab}}$ satisfies the following pseudoidentity:

$$stxyz = sxyzs(sxzs)^{\omega-1}stxz. \tag{2}$$

Now, let subsets A and B of a semigroup $S\in\mathbf{CS}\cap\overline{\mathbf{Ab}}$ be given and let m be a positive integer such that $\mathscr{P}'(S)$ satisfies the pseudoidentity $x^{\omega}=x^{m}$. Then, for every $k\geq 2$ and $l\in\mathbb{Z}$, taking $s\in A^k$, $t\in B$, $x\in A^{k-1}$, $y\in A^{\omega-l}$, and $z\in A$, $stxyz$ describes an arbitrary element of $A^kBA^{\omega-l+k}$, whereas $sxyzs(sxzs)^{\omega-1}stxz$ lies in

$$A^{3k+k-1+1+\omega-l+(\omega-1)(2k+k-1+1)}BA^{k-1+1} = A^{\omega-l+k}BA^{k}.$$

By the pseudoidentity (2), it follows that $A^kBA^{\omega-l+k}\subseteq A^{\omega-l+k}BA^k$, and equality follows by symmetry. This proves the folowing result which extends [11, Proposition 11.9.8].

Proposition 5.20. *The pseudovariety $\mathcal{P}(\mathbf{CS}\cap\overline{\mathbf{Ab}})$ satisfies the pseudoidentity $x^kyx^{\omega+l} = x^{\omega+l}yx^k$ for every $k,l\in\mathbb{Z}$ with $k\geq 2$.*$\square$

On the other hand, by Theorem 5.9, we have

$$\mathcal{P}(\mathbf{CS}\cap\overline{\mathbf{Ab}}) \subseteq \overline{\mathbf{Ab}},$$

a result which is also a corollary of Proposition 5.20. According to [11, Exercise 11.9.9], the restriction $k\geq 2$ may not be removed in the above statement. The best upper bound known for the power of the pseudovariety $\mathcal{V}\{K_p\}$ is the one given by Proposition 5.20 together with Theorem 5.9 which gives the pseudoidentity $x^{\omega+p}=x^{\omega}$.

Proposition 2.3 has recently been extended as follows.

Proposition 5.21 [27]. $\mathcal{P}\mathbf{RO}\subseteq\mathcal{B}\mathbf{ReG}$.

In case the subgroups are Abelian, Colaço [27] also characterized which $\mathbf{V} \subseteq \mathbf{RO} \cap \overline{\mathbf{Ab}}$ have $\mathcal{P}\mathbf{V} \subseteq \mathcal{D}\mathbf{S}$ and these results were crucial to give the following characterization of the exponent for pseudovarieties of monoids. To state it, we need another two families of semigroups. For $n \geq 2$, the semigroup S_n is given by the set

$$S_n = \{e_1, \ldots, e_n\} \times Z_n \cup \{f_1, \ldots, f_n\},$$

where $\{e_1, \ldots, e_n\}$ is a left-zero subsemigroup, Z_n is a cyclic group with generator a, $\{e_1, \ldots, e_n\} \times Z_n$ is a rectangular group subsemigroup, the f_i are left zeros, and $(e_i, a^j)f^k = f_{i+j \,(\bmod n)}$. Also for $n \geq 2$, T_n is the monoid defined by the following presentation:

$$T_n = \langle a, b; a^n = 1,\, ab = b,\, b^2 = 0 \rangle.$$

Theorem 5.22 [27]. *Let* $\mathbf{V}$ *be a non-commutative pseudovariety of monoids. Then:*

a) $\mathbf{V}$ *has exponent 0 if and only if* $\mathbf{V} = \mathbf{M}$*;*

b) $\mathbf{V}$ *has exponent 1 if and only if* $B_2^1 \in \mathbf{V}$*;*

c) $\mathbf{V}$ *has exponent 2 if and only if* $B_2^1 \notin \mathbf{V}$ *and* $\mathbf{V}$ *contains a non-Abelian group or one of the monoids* Y^1*,* S_n^1*,* $(S_n^1)\rho$*, or* T_n $(n \geq 2)$*;*

d) $\mathbf{V}$ *has exponent 3 if and only if* $\mathbf{V}$ *contains none of the monoids mentioned in (c).*

6. Extensions

By a *relational morphism* μ from a semigroup S to a semigroup T, we mean a subsemigroup of $S \times T$ which, as a relation, has domain S. A $\mathbf{V}$*-pointlike* subset of a semigroup S is a subset X of S such that, for every relational morphism $\mu : S \to T$ with $T \in \mathbf{V}$, there exists $t \in T$ such that $X \times \{t\} \subseteq \mu$. We denote by $\mathscr{P}_\mathbf{V}(S)$ the set of all $\mathbf{V}$-pointlike subsets of S which constitutes a subsemigroup of $\mathscr{P}(S)$. For two pseudovarieties $\mathbf{V}$ and $\mathbf{W}$, we let

$$\mathcal{P}_\mathbf{V}\mathbf{W} = \mathcal{V}\{\mathscr{P}_\mathbf{V}(S) : S \in \mathbf{W}\} = \mathcal{HS}\{\mathscr{P}_\mathbf{V}(S) : S \in \mathbf{W}\}.$$

For instance, algorithms for the computation of $\mathscr{P}_\mathbf{G}(S)$ and $\mathscr{P}_\mathbf{A}(S)$ have been obtained respectively by Ash [21] and Henckell [30, 31]. So, a natural question that comes up is to study the operators $\mathcal{P}_\mathbf{G}$ and $\mathcal{P}_\mathbf{A}$. Colaço [27] has started the study of the operator $\mathcal{P}_\mathbf{G}$, leading to some rather interesting results which we now briefly describe.

Another semigroup which has played a role in various contexts is the one described by the following presentation:

$$A_2 = \langle a, b; a^2 = a,\, b^2 = 0,\, aba = a,\, bab = b \rangle.$$

Consider also the semigroup presented by

$$Z = \langle e, f; e^2 = e,\, f^2 = f,\, fe = 0 \rangle.$$

Extending work of Ash [20] for the semilattice case, Margolis, Birget and Rhodes [40] showed that $\mathbf{B} \textcircled{m} \mathbf{G} = [\![(ef)^2 = ef]\!]$. It follows that, for every subpseudovariety of $\mathbf{B}$,

$\mathbf{V} ⓜ \mathbf{G} = \mathcal{E}\mathbf{V}$. On the other hand, Colaço and the author have shown that $\mathcal{D}\mathbf{S} * \mathbf{G} = \mathcal{E}\mathcal{D}\mathbf{S}$, while from results of Ash [32] and Jones and Trotter [36][1] it follows that $\mathcal{D}\mathbf{S} * \mathbf{G} = \mathcal{D}\mathbf{S} ⓜ \mathbf{G}$.[2] This gives a hint of what is going on with the operator $\mathcal{P}_{\mathbf{G}}$.

The $\mathbf{G}$-*exponent* of a pseudovariety $\mathbf{V}$ is the least integer $n \geq 0$ such that $\mathcal{P}_{\mathbf{G}}^n\mathbf{V} = \mathbf{M}$ if such an integer exists, where $\mathcal{P}_{\mathbf{G}}^0\mathbf{V} = \mathbf{V}$. If there is no such integer, we say that the exponent of $\mathbf{V}$ is infinite.

Theorem 6.1 [27]. *Let $\mathbf{V}$ be a pseudovariety of monoids. Then*

a) $\mathbf{V}$ has $\mathbf{G}$-exponent 0 if and only if $\mathbf{V} = \mathbf{M}$;

b) $\mathbf{V}$ has $\mathbf{G}$-exponent 1 if and only if $\mathbf{V} \neq \mathbf{M}$ and $A_2^1 \in \mathbf{V}$;

c) $\mathbf{V}$ has $\mathbf{G}$-exponent 2 if and only if both $Z^1 \in \mathbf{V}$ and $\mathbf{V} \subseteq \mathcal{E}\mathcal{D}\mathbf{S}$, or both $\mathbf{V} \subseteq \mathcal{E}\mathbf{CR}$ and $\mathbf{V} \nsubseteq \mathcal{E}\mathcal{M}\mathbf{NB}$;

d) $\mathbf{V}$ has $\mathbf{G}$-exponent 3 if and only if $\mathbf{V} \nsubseteq \mathcal{E}\mathbf{Sl}$ and $\mathbf{V} \subseteq \mathcal{E}\mathcal{M}\mathbf{NB}$;

e) $\mathbf{V}$ has infinite $\mathbf{G}$-exponent if and only if $\mathbf{V} \subseteq \mathcal{E}\mathbf{Sl}$.

Another possible direction for extending the theory of power pseudovarieties is to look at subsets of a semigroup S as formal series in the elements of S with coefficients in the Boolean semiring $B = \{0, 1\}$ (where a coefficient of 1 indicates that the element belongs to the subset). Thus, $\mathscr{P}(S)$ is just the multiplicative semigroup of the semigroup semiring $B[[S]]$. By considering semirings other than B, one may develop similar results. This has been done in part by Blanchard [24, 25].

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[1]The result of Jones and Trotter (namely the *locality* of $\mathbf{DS}$) depends on a result of Jones and Pustejovsky [35] whose proof L. Teixeira has shown to be flawed. Recently, Steinberg [57] has announced a proof of the result previsouly stated by Jones and Pustejovsky.

[2]This equality also follows from $\mathcal{D}\mathbf{S} * \mathbf{G} = \mathcal{E}\mathcal{D}\mathbf{S}$ since it is easy to show that $\mathcal{D}\mathbf{S} * \mathbf{G} \subseteq \mathcal{D}\mathbf{S} ⓜ \mathbf{G} \subseteq \mathcal{E}\mathcal{D}\mathbf{S}$.

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