Open Covers and Compactness

Suppose (X, d) is a metric space.

Definition

Let $E \subseteq X$. An *open cover* of E is a collection $\{G_\alpha : \alpha \in I\}$ of open subsets of X such that $E \subseteq \bigcup_{\alpha \in I} G_\alpha$

Definition

A subset K of X is *compact* if every open cover contains a *finite* subcover.

In other words if $\{G_\alpha : \alpha \in I\}$ is a collection of open subsets of X with $K \subseteq \bigcup_{\alpha \in I} G_\alpha$ then there is a finite set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset I$ such that

$$K \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n}$$

Examples

Examples of Compact Sets:

- Every finite set is compact.
- Any closed interval $[a, b]$ in $\mathbb{R}^1$.

Examples of Non-Compact Sets:

- $\mathbb{Z}$ in $\mathbb{R}^1$.
- Any open interval (a, b) in $\mathbb{R}^1$.
- $\mathbb{R}^1$ as a subset of $\mathbb{R}^1$.

Relative Compactness

Theorem

Suppose (X, d) is a metric space and $K \subseteq Y \subseteq X$. Then K is a compact subset of (X, d) if and only if K is a compact subset of (Y, d).

So unlike with closed and open sets, a set is "compact relative a subset Y" if and only if it is compact relative to the whole space.

Compact Subsets are Closed

Theorem

Compact subsets of a metric space are closed.

Closed Subsets and Compactness

Theorem

Closed subsets of compact sets are compact.

Corollary

If F is closed and K is compact then $F \cap K$ is compact.

Intersection of Compact Sets

Theorem

If $\{K_\alpha : \alpha \in I\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha : \alpha \in I\}$ is non-empty then $\bigcap_{\alpha \in I} K_\alpha$ is nonempty.

Corollary

If $\{K_n : n \in \mathbb{N}\}$ is a sequence of nonempty compact sets such that $K_n \supseteq K_{n+1}$ (for $n = 1, 2, 3, \dots$) then $\bigcap_1^\infty K_n$ is not empty.

Limit Points in Compact Sets

Theorem

Every infinite subset of a compact set K has a limit point in K.

Intersection of k-cells

Theorem

If $\{I_n : n \in \mathbb{N}\}$ is a sequence of nonempty intervals in $\mathbb{R}^1$ such that $I_n \supseteq I_{n+1}$ (for $n = 1, 2, 3, \ldots$) then $\bigcap_1^\infty I_n$ is not empty.

Theorem

Let k be a positive integer. If $\{I_n : n \in \mathbb{N}\}$ is a sequence of nonempty k-cells such that $I_n \supseteq I_{n+1}$ (for $n = 1, 2, 3, \ldots$) then $\bigcap_1^\infty I_n$ is not empty.

k-Cells are Compact

Theorem

Every k-cell is compact.

Closed and Bounded Subsets of $\mathbb{R}^k$

Theorem

If $E \subseteq \mathbb{R}^k$ then the following are equivalent:

(a) *E is closed and bounded.*

(b) *E is compact.*

(c) *Every infinite subset of E has a limit point in E.*

Corollary (Weierstrass)

Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Separated Sets

Definition

Two subsets A, B of a metric space X are said to be *separated* if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty. I.e. if no point of A lies in the closure of B and no point of B lies in the closure of A.

A set $E \subseteq X$ is said to be *connected* if E is not the union of two nonempty separated sets.

Note that while any two separated sets are disjoint, not all disjoint sets are separate.

Consider $[0,1]$ and $(1,2)$. $[0,1] \cap (1,2) = \emptyset$ but $1 \in [0,1]$ and 1 is a limit point of $(1,2)$.

Connected Subsets of $\mathbb{R}^1$

Theorem

A subset E of the real line $\mathbb{R}^1$ is connected if and only if it has the following property: If $x \in E, y \in E$ and $x < z < y$ then $z \in E$.