Differential Equations [1]

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PART I: Ordinary Differential Equations

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1 First order ODE

1.1 Introduction

An **Ordinary differential equation** (ODE) is an equation involving an unknown function and its derivatives with respect to an independent variable x:

$$F(x, y, y^{(1)}, \dots y^{(k)}) = 0.$$

Here, y is the unknown function, x is the independent variable and $y^{(j)}$ represents the j-th derivative of y. We shall also denote

$$y' = y^{(1)}, \quad y'' = y^{(2)}, \quad y''' = y^{(3)}.$$

Thus, a first order ODE is of the form

$$F(x, y, y') = 0. \tag{$*$}$$

Sometimes the above equation can be put in the form:

$$y' = f(x, y). \tag{1}$$

By a **solution** of $(*)$ we mean a function $y = \varphi(x)$ defined on an interval $I := (a, b)$ which is differentiable and satisfies $(*)$, i.e.,

$$F(x, \varphi(x), \varphi'(x)) = 0, \quad x \in I.$$

Example 1.1.

$$y' = x.$$

Note that, for every constant C, $y = x^2/2 + C$ satisfies the DE for every $x \in \mathbb{R}$. $\diamondsuit$

The above simple example shows that a DE can have more than one solution. In fact, we obtain a family of parabolas as **solution curves**. But, if we require the *solution curve* to pass through certain specified point then we may get a unique solution. In the above example, if we demand that

$$y(x_0) = y_0$$

for some given x_0, y_0, then we must have

$$y_0 = \frac{x_0^2}{2} + C$$

so that the constant C must be

$$C = y_0 - \frac{x_0^2}{2}.$$

Thus, the solution, in this case, must be

$$y = \frac{x^2}{2} + y_0 - \frac{x_0^2}{2}.$$

1.2 Direction Field and Isoclines

Suppose $y = \varphi(x)$ is a solution of DE (1). Then this curve is also called an **integral curve** of the DE. At each point on this curve, the tangent must have the slope $f(x, y)$. Thus, the DE prescribes a direction at each point on the integral curve $y = \varphi(x)$. Such directions can be represented by small line segments with arrows pointing to the direction. The set of all such directed line segments is called the **direction field** of the DE.

The set of all points in the plane where $f(x, y)$ is a constant is called an **isocline**. Thus, the family of isoclines would help us locating integral curves geometrically.

Isoclines for the DE: $y' = x + y$ are the straight lines $x + y = C$.

1.3 Initial Value Problem

An equation of the form

$$y' = f(x, y) \tag{1}$$

together with a condition of the form the form

$$y(x_0) = y_0 \tag{2}$$

is called an **initial value problem**. The condition (2) is called an **initial condition**.

THEOREM 1.2. *Suppose f is defined in an open rectangle $R = I \times J$, where I and J are open intervals, say $I = (a, b)$, $J = (c, d)$:*

$$R := \{(x, y) : a < x < b, \quad c < y < d\}.$$

If f is continuous and has continuous partial derivative $\dfrac{\partial f}{\partial y}$ in R, then for every $(x_0, y_0) \in R$, there exists a unique function $y = \varphi(x)$ defined in an interval $(x_0 - h, x_0 + h) \subseteq (a, b)$ which satisfies $(1) - (2)$.

Remark 1.3. The conditions prescribed are sufficient conditions that guarantee the existence and uniqueness of a solution for the initial value problem. They are not necessary conditions. A unique solution for the initial value problem can exist without the prescribed conditions on f as in the above theorem. $\diamondsuit$

- The condition (2) in Theorem 1.2 is called an **initial condition**, the equation (1) together with (2) is called an **initial value problem**.
- A solution of (1) the form

  $$y = \varphi(x, C),$$

  where C is an arbitrary constant varying in some subset of $\mathbb{R}$, is called a **general solution** of (1).

- A solution y for a particular value of C is called a **particular solution** of (1).
- If general solutions of (1) are given implicitly in the form

$$u(x, y, C) = 0$$

  arbitrary constant C, then the above equation is called the **complete integral** of (1).
- A complete integral for a particular value of C is called a **particular integral** of (1).

Remark 1.4. Under the assumptions of Theorem 1.2, if $x_0 \in I$, then existence of a solution y for (1) is guaranteed in some neighbourhood $I_0 \subseteq I$ of x_0, and it satisfies the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt.$$

A natural question would be:

> Is the family of all solutions of (1) defined on I_0 a one-parameter family, so that any two solutions in that family differ only by a constant?

It is known that for a general nonlinear equation (1), the answer is nt in affirmative. However, for **linear equations** the answer is in affirmative. $\diamondsuit$

1.4 Linear ODE

If f depends of y in a linear fashion, then the equation (1) is called a **linear DE**. A general form of the linear first order DE is:

$$y' + p(x)y = q(x). \tag{3}$$

Here is a procedure to arrive at a solution of (3):

Assume first that there is a solution for (3) and that after multiplying both sides of (3) by a differentiable function $\mu(x)$, the LHS is of the $(\mu(x)y)'$. Then(3) will be converted into:

$$(\mu(x)y)' = \mu(x)q(x)$$

so that

$$\mu(x)y = \int q(x)dx + C.$$

Thus, μ must be chosen in such a manner that

$$\mu' y + \mu y' = \mu(y' + py).$$

Therefore, we must have

$$\mu' y = \mu p y, \quad \text{i.e.,} \quad \mu' = \mu p, \quad \text{i.e.,} \quad \frac{d\mu}{\mu} = p dx,$$

i.e.,

$$\mu(x) := e^{\int p(x)dx}.$$

Thus, y takes the form

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)q(x)dx + C\right], \quad \mu(x) := e^{\int p(x)dx}. \tag{4}$$

It can be easily seen that the function y defined by (4) satisfies the DE (3). Thus **existence** of a solution for (3) is proved for continuous functions p and q.

Suppose there are two functions φ and ψ which satisfy (3). Then $\chi(x) := \varphi(x) - \psi(x)$ would satisfy

$$\chi'(x) + p(x)\chi(x) = 0.$$

Hence, using the arguments in the previous paragraph, we obtain

$$\chi(x) = C\mu(x)^{-1}$$

for some constant C.

Now, if $\varphi(x_0) = y_0 = \psi(x_0)$, then we must have $\chi(x_0) = 0$ so that $C\mu(x)^{-1} = 0$. Hence, we obtain $C = 0$ and hence, $\varphi = \psi$. Thus, we have proved the existence and uniqueness for the linear DE only by assuming that p and q are continuous.

Example 1.5.

$$y' = x + y.$$

Then, $\mu = e^{-\int dx} = e^{-x}$ and hence,

$$y = e^x\left[\int e^{-x}xdx + C\right] = e^x\left[-xe^{-x} + \int e^{-x}dx + C\right].$$

Thus,

$$y = e^x\left[-xe^{-x} - e^{-x} + C\right] = -x - 1 + Ce^x.$$

$$y(0) = 0 \implies 0 = -1 + C \implies C = 1.$$

Hence,

$$y = -x - 1 + e^x.$$

Note that

$$y' = -1 + e^x = -1 + (x + y + 1) = x + y.$$

$\diamondsuit$

1.5 Equations with Variables Separated

If $f(x,y)$ in (1) is of the form

$$f(x,y) = f_1(x)f_2(y)$$

for some functions f_1, f_2, then we say that (3) is an **equation with separated variables**. In this case (3) takes the form:

$$y' = f_1(x)f_2(y);$$

equivalently,

$$\frac{y'}{f_2(y)} = f_1(x),$$

assuming that $f_2(y)$ is not zero at all points in the interval of interest. Hence, in this case, a **general solution** is given **implicitly** by

$$\int \frac{dy}{f_2(y)} = \int f_1(x)dx + C.$$

Example 1.6.

$$y' = xy.$$

Equivalently,

$$\frac{dy}{y} = xdx.$$

Hence,

$$\log|y| = \frac{x^2}{2} + C,$$

i.e.,

$$y = C_1 e^{x^2/2}.$$

Note that

$$y = C_1 e^{x^2/2} \quad \Longrightarrow \quad y' = C_1 \left(e^{x^2/2}x\right) = xy.$$

$\diamondsuit$

An equation with separated variables can also be written as

$$M(x)dx + N(y)dy = 0.$$

In this case, solution is implicitly defined by

$$\int M(x)dx + \int N(y)dy = 0. \tag{5}$$

Equation of the form

$$M_1(x)N_1(y)dx + M_2(x)N_2(y)dy = 0 \tag{6}$$

can be brought to the form (5): After dividing (6) by $N_1(y)M_2(x)$ we obtain

$$\frac{M_1(x)}{M_2(x)}dx + \frac{N_2(y)}{N_1(y)}dy = 0.$$

1.6 Homogeneous equations

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **homogeneous of degree** n if

$$f(\lambda x, \lambda y) = \lambda^n f(x, y) \quad \forall\, \lambda \in \mathbb{R}$$

for some $n \in \mathbb{N}$.

The differential equation (1) is called a **homogeneous equation** if f is homogeneous of degree 0, i.e., if

$$f(\lambda x, \lambda y) = f(x, y) \quad \forall\, \lambda \in \mathbb{R}.$$

Suppose (1) is a homogeneous equation. Then we have

$$y' = f(x, y) = f(\frac{x}{x}, \frac{y}{x}) = f(1, u), \quad u := \frac{y}{x}.$$

Now,

$$u = \frac{y}{x} \quad \Longrightarrow \quad ux = y \Longrightarrow \quad u + x\frac{du}{dx} = y' = f(1, u).$$

Thus,

$$\frac{du}{f(1, u) - u} = \frac{dx}{x}$$

and hence, u and therefore, y is implicitly defined by

$$\int \frac{du}{f(1, u) - u} = \int \frac{dx}{x} + C.$$

1.7 Exact Equations

Suppose (1) is of the form

$$M(x, y)dx + N(x, y)dy = 0, \tag{7}$$

where M and N are such that there exists $u(x, y)$ with continuous first partial derivatives satisfying

$$M(x, y) = \frac{\partial u}{\partial x}, \quad N(x, y) = \frac{\partial u}{\partial y}. \tag{8}$$

Then (7) takes the form

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0;$$

equivalently,

$$du = 0.$$

Then the general solution is implicitly defined by

$$u(x, y) = C.$$

Equation (7) with M and N satisfying (8) is called an **exact differential equation**.

Note that, in the above, if there exists $u(x,y)$ with continuous second partial derivatives $\dfrac{\partial^2 u}{\partial x \partial y}$ and $\dfrac{\partial^2 u}{\partial y \partial x}$, then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

In fact it is a sufficient condition of (7) to be an exact differential equation.

THEOREM 1.7. *Suppose M and N are continuous and have continuous first partial derivatives $\dfrac{\partial M}{\partial y}$ and $\dfrac{\partial N}{\partial x}$ in $I \times J$, and*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Then the equation (7) is exact, and in that case the complete integral of (7) is given by

$$\int_{x_0}^{x} M(x,y)dx + \int_{y_0}^{y} N(x_0,y)dy = C.$$

Proof. Note that for any differentiable function $g(y)$,

$$u(x,y) := \int_{x_0}^{x} M(x,y)dx + g(y)$$

satisfies $\dfrac{\partial u}{\partial x} = M(x,y)$. Then

$$\frac{\partial u}{\partial y} = \int_{x_0}^{x} \frac{\partial M}{\partial y} dx + g'(y) = \int_{x_0}^{x} \frac{\partial N}{\partial x} dx + g'(y) = N(x,y) - N(x_0,y) + g'(y).$$

Thus,

$$\frac{\partial u}{\partial y} = N \iff g'(y) = N(x_0,y) \iff g(y) = \int_{y_0}^{y} N(x_0,y)dy.$$

Thus, taking

$$g(y) = \int_{y_0}^{y} N(x_0,y)dy \quad \text{and} \quad u(x,y) := \int_{x_0}^{x} M(x,y)dx + g(y)$$

we obtain (8), and the complete integral of (7) is given by

$$\int_{x_0}^{x} M(x,y)dx + \int_{y_0}^{y} N(x_0,y)dy = C.$$

□

Example 1.8.

$$y \cos xy dx + x \cos xy dy = 0.$$

$$\varphi(x,y) = \sin xy \implies \frac{\partial \varphi}{\partial x} = x \cos xy \quad \text{and} \quad \frac{\partial \varphi}{\partial y} = x \cos xy.$$

Hence, $\sin xy = C$. Also,

$$y \cos xy dx + x \cos xy dy = 0 \iff y' = -\frac{y}{x} \iff \frac{dx}{x} + \frac{dy}{y} = 0.$$

Hence, $\log |xy| = C$. ♢

Example 1.9.

$$\frac{2x}{y^3}dx + \frac{y^2 - 3x^2}{y^4}dy = 0.$$

In this case

$$\frac{\partial M}{\partial y} = -\frac{6x}{y^4} = \frac{\partial N}{\partial x}.$$

Hence, the given DE is exact, and u is give by

$$u(x,y) = \int M dx + \int N(0,y)dy = \frac{x^2}{y^3} - \frac{1}{y},$$

so that the complete integral is given by $u(x,y) = C$. $\diamondsuit$

1.8 Equations reducible to homogeneous or variable separable or linear or exact form

1.8.1 Reducible to homogeneous or variable separable form

Note that the function

$$f(x,y) = \frac{ax + by + c}{a_1x + b_1y + c_1}$$

is not homogeneous if either $c \neq 0$ or $c_1 \neq 0$, and in such case,

$$\frac{dy}{dx} = f(x,y) \tag{1}$$

is not homogeneous. We shall convert this equation into a homogenous equation in terms a variables: Consider the change of variables:

$$X = x - h, \quad Y = y - k.$$

Then

$$ax + by + c = a(X + h) + b(Y + k) + c = aX + bY + (ah + bk + c),$$

$$a_1x + b_1y + c_1 = a_1(X + h) + b_1(Y + k) + c_1 = a_1X + b_2Y + (a_1h + b_1k + c_1).$$

There are two cases:

Case(i): $det \begin{pmatrix} a & b \\ a_1 & b_1 \end{pmatrix} \neq 0.$

In this case there exists a unique pair (h,k) such that

$$ah + bk + c = 0 \tag{2}$$

$$a_1h + b_1k + c_1 \tag{3}$$

are satisfied. Hence, observing that

$$\frac{dY}{dX} = \frac{dY}{dy}\frac{dy}{dx}\frac{dx}{dX} = \frac{dy}{dx},$$

the equation (1) takes the form

$$\frac{dY}{dX} = \frac{aX + bY}{a_1 X + b_1 Y}.$$

This is a homogeneous equation. If $Y = \varphi(X)$ is a solution of this homogeneous equation, then a solution of (1) is given by

$$y = k + \varphi(x - h).$$

Case(ii): $det \begin{pmatrix} a & b \\ a_1 & b_1 \end{pmatrix} = 0$. In this case either

$$a_1 = \alpha a, \quad b_1 = \alpha b \quad \text{ for some } \alpha \in \mathbb{R}$$

or

$$a_1 = \beta a_1,\ b = \beta b_1 \text{ for some } \beta \in \mathbb{R}.$$

Assume that $a_1 = \alpha a$ and $b_1 = \alpha b$ for some $\alpha \in \mathbb{R}$. Then, (1) takes the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a_1 x + b_1 y + c_1} = \frac{ax + by + c}{\alpha(ax + by) + c_1}.$$

Taking $z = ax + by$, we obtain

$$\frac{dz}{dx} = a + b\frac{dy}{dx} = a + b\left(\frac{z + c}{\alpha(z + c_1}\right).$$

This is an equation in variable separable form.

Example 1.10.

$$\frac{dy}{dx} = \frac{2x + y - 1}{4x + 2y + 5}.$$

Taking $z = 2x + y$,

$$\frac{dz}{dx} = 2 + \frac{dy}{dx} = 2 + \frac{z - 1}{2z + 5} \iff \frac{dz}{dx} = \frac{5z + 9}{2z + 5},$$

i.e.,

$$\frac{2z + 5}{5z + 9} dz = dx.$$

Note that

$$\frac{2z + 5}{5z + 9} = \left(\frac{1}{5}\right)\frac{10z + 25}{5z + 9} = \left(\frac{1}{5}\right)\frac{2(5z + 9) + 7}{5z + 9} = \left(\frac{2}{5}\right) + \left(\frac{7}{5}\right)\frac{1}{5z + 9}$$

$$\int \frac{2z + 5}{5z + 9} dz = \inf dx \iff \frac{2z}{5} + \frac{7}{25}\log|5z + 9| = x + 9$$

$$\iff \frac{2(2x + y)}{5} + \frac{7}{25}\log|5(2x + y) + 9| = x + 9$$

Thus, the solution y is given by

$$\frac{4x + 2y}{5} + \frac{7}{25}\log|10x + 5y + 9| = x + 9.$$

$\diamondsuit$

1.8.2 Reducible to linear form

Bernauli's equation:

$$y' + p(x)y = q(x)y^n.$$

Write it as

$$y^{-n}y' + p(x)y^{-n+1} = q(x).$$

Taking $z = y^{-n+1}$,

$$\frac{dz}{dx} = (-n+1)y^{-n}\frac{dy}{dx} = (-n+1)[-p(x)z + q(x)],$$

i.e.,

$$\frac{dz}{dx} - (-n+1)p(x)z = (-n+1)q(x).$$

Hence,

$$z = \frac{1}{\mu(x)}\left(\int \mu(x)(-n+1)q(x)dx + C\right), \quad \mu(x) = e^{(-n+1)\int p(x)dx}.$$

Example 1.11.

$$\frac{dy}{dx} + xy = x^3y^3.$$

Here, $n = 3$ so that $-n+1 = -2$ and

$$\mu(x) = e^{(-n+1)\int p(x)dx} = e^{-2\int xdx} = e^{-x^2}.$$

$$\begin{aligned} z &= \frac{1}{\mu(x)}\left(\int \mu(x)(-n+1)q(x)dx + C\right) = e^{x^2}\left(\int -2e^{-x^2}x^3dx + C\right) \\ &= -2e^{x^2}\left(\int e^{-x^2}x^3dx - C/2\right). \end{aligned}$$

Gives:

$$(x^2 + 1 + Ce^{x^2})y^2 = 1.$$

$\diamondsuit$

1.8.3 Reducible to exact equations

Suppose $M(x,y)$ and $N(x,y)$ are functions with continuous partial derivatives $\frac{\partial M}{\partial x}, \frac{\partial N}{\partial x}, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$. Consider the differential equation

$$M(x,y)dx + N(x,y)dy = 0.$$

Recall that it is an exact equation if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Suppose the equation is not exact. Then we look for a function $\mu := \mu(x)$ such that

$$\mu(x)[M(x,y)dx + N(x,y)dy] = 0 \qquad (*)$$

is exact. So, requirement on μ should be

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N), \; i.e., \mu\frac{\partial M}{\partial y} = \mu\frac{\partial N}{\partial x} + \mu' N$$

$$\iff \qquad \frac{\mu'}{\mu} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right).$$

Thus:

> If $\varphi := \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone, then the above differential equation for μ can be solved and with the resulting $\mu := e^{\int \varphi dx}$ the equation $(*)$ is exact equation.

Similarly, looking for a function $\tilde{\mu} = \tilde{\mu}(y)$ such that

$$\tilde{\mu}(x)[M(x,y)dx + N(x,y)dy] = 0 \qquad (**)$$

becomes exact, we arrive at the equation

$$\frac{\tilde{\mu}'(y)}{\tilde{\mu}(y)} = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

Hence, we can make the following statement:

> If $\psi := \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of y alone, then the above differential equation for μ can be solved and with the resulting $\mu := e^{\int \psi dx}$ the equation $(**)$ is exact equation.

Definition 1.12. Each of the functions $\mu(x)$ and $\tilde{\mu}(y)$ in the above discussion, if exists, is called an **integrating factor**. ♢

Example 1.13.

$$(y + xy^2)dx - xdy = 0.$$

Note that $\frac{\partial M}{\partial y} = 1 + 2xy, \quad \frac{\partial N}{\partial x} = -1,$

$$\varphi := \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{(1+2xy)+1}{-x} = \frac{2(1+xy)}{-x}.$$

$$\psi = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{-2(1+xy)}{y(1+xy)} = -\frac{2}{y}.$$

Thus,

$$\tilde{\mu} := e^{\int \frac{-2}{y} dy} = -\frac{1}{y^2}$$

is an integrating factor, i.e.,

$$-\frac{1}{y^2}[(y+xy^2)dx - xdy] = 0 \iff \left(-\frac{1}{y} - x\right)dx - \frac{x}{y^2}dy = 0$$

is an exact equation. Then

$$u = \int M dx + \int N(0,y)dy = \int \left(-\frac{1}{y} - x\right)dx = -\frac{x}{y} - \frac{x^2}{2}.$$

Thus the complete integral is given by $\frac{x}{y} + \frac{x^2}{2} = C.$ ♢

2 Second and higher order linear ODE

Second order linear ODE is of the form

$$y'' + a(x)y' + b(x)y = f(x) \tag{1}$$

where $a(x), b(x), f(x)$ are functions defined on some interval I. The equation (1) is said to be

1. **homogeneous** if $f(x) = 0$ for all $x \in I$, and
2. **non-homogeneous** of $f(x) = 0$ for some $x \in I$.

THEOREM 2.1. (Existence and uniqueness) *Suppose $a(x), b(x), f(x)$ are continuous functions (defined on some interval I). Then for every $x_0 \in I$, $y_0 \in \mathbb{R}$, $z_0 \in \mathbb{R}$, there exists a unique solution y for (1) such that*

$$y(x_0) = y_0, \quad y'(x_0) = z_0.$$

2.1 Second order linear homogeneous ODE

Consider second order linear homogeneous ODE:

$$y'' + a(x)y' + b(x)y = 0. \tag{2}$$

Note that:

- If y_1 and y_2 are solutions of (2), then for any $\alpha, \beta \in \mathbb{R}$, the function $\alpha y_1 + \beta y_2$ is also a solution of (2).

Definition 2.2. Let y_1 and y_2 be functions defined on an interval I.

1. y_1 and y_2 are said to be **linearly dependent** if there exists $\lambda \in \mathbb{R}$ such that either $y_1(x) = \lambda y_2(x)$ or $y_2(x) = \lambda y_1(x)$; equivalently, there exists $\alpha, \beta \in \mathbb{R}$ with atleast one of them nonzero, such that
$$\alpha y_1(x) + \beta y_2(x) = 0 \quad \forall\, x \in I.$$
2. y_1 and y_2 are said to be **linearly independent** if they are not linearly dependent, i.e. for $\alpha, \beta \in \mathbb{R}$,
$$\alpha y_1(x) + \beta y_2(x) = 0 \quad \forall\, x \in I \quad \Longrightarrow \quad \alpha = 0,\, \beta = 0.$$

$\diamondsuit$

We shall prove:

THEOREM 2.3. *The following hold.*

1. *The differential equation (2) has two linearly independent solutions.*

2. *If y_1 and y_2 are linearly independent solutions of (2), then every solution y of (2) can be expressed as*

$$y = \alpha y_1 + \beta y_2$$

for some α, $\beta \in \mathbb{R}$.

Definition 2.4. Let y_1 and y_2 be differentiable functions (on an interval I). Then the function

$$W(y_1, y_2)(x) := \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

is called the **Wronskian** of y_1, y_2. $\diamondsuit$

Once the functions y_1, y_2 are fixed, we shall denote $W(y_1, y_2)(x)$ by $W(x)$.

Note that:

- If y_1 and y_2 are linearly dependent, then $W(x) = 0$ for all $x \in I$.

Equivalently:

- If $W(x_0) \neq 0$ for some $x_0 \in I$, then y_1 and y_2 are linearly independent.

THEOREM 2.5. *Consider a nonsingular matrix $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$. Let $x_0 \in I$. Let y_1 and y_2 be unique solutions of (2) satisfying the conditions*

$$\begin{array}{ll} y_1(x_0) = a_1 & y_2(x_0) = b_1 \\ y_1'(x_0) = a_2 & y_2'(x_0) = b_2 \end{array}.$$

Then y_1 and y_2 are linearly independent solutions of (2).

Proof. Since $A = W(x_0)$ and $\det(A) \neq 0$, the proof follows from the earlier observation. $\square$

LEMMA 2.6. *Let y_1 and y_2 be solutions of (2) and $x_0 \in I$. Then*

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} a(t)dt}.$$

In particular, if y_1 and y_2 are solutions of (2), then

$$W(x_0) = 0 \text{ at some point } x_0 \iff W(x) = 0 \text{ at every point } x \in I.$$

Proof. Since y_1 and y_2 are solutions of (2), we have

$$y_1'' + a(x)y_1' + b(x)y_1 = 0,$$

$$y_2'' + a(x)y_2' + b(x)y_2 = 0.$$

Hence,

$$(y_1y_2'' - y_2y_1'') + a(x)(y_1y_2' - y_2y_1') = 0.$$

Note that

$$W = y_1y_2' - y_2y_1', \quad W' = y_1y_2'' - y_2y_1''.$$

Hence

$$W' + a(x)W = 0.$$

Therefore,

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} a(t)dt}.$$

□

THEOREM 2.7. *Let y_1 and y_2 be solutions of (2) and $x_0 \in I$. Then*

$$y_1 \text{ and } y_2 \text{ are linearly independent,} \iff W(x) \neq 0 \text{ for every } x \in I.$$

Proof. We have already observed that if $W(x_0) = 0$ for some $x_0 \in I$, then y_1 and y_2 are linearly independent. Hence, it remains to prove that if y_1 and y_2 are linearly independent, then $W(x) \neq 0$ for every $x \in I$.

Suppose $W(x_0) = 0$ for some $x_0 \in I$. Then by the Lemma 2.6, $W(x) = 0$ for every $x \in I$,i.e.,

$$y_1y_2' - y_2y_1' = 0 \quad \text{on } I.$$

Let $I_0 = \{x \in I : y_1(x) \neq 0\}$. Then we have

$$\frac{y_1y_2' - y_2y_1'}{y_1^2} = 0 \quad \text{on } I_0,$$

i.e.,

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = 0 \quad \text{on } I_0.$$

Hence, there exists $\lambda \in \mathbb{R}$ such that

$$\frac{y_2}{y_1} = \lambda \quad \text{on } I_0.$$

Hence, $y_2 = \lambda y_1$ on I, showing that y_1 and y_2 are linearly dependent. □

THEOREM 2.8. *Let y_1 and y_2 be linearly independent solutions of (2). Then every solution y of (2) can be expressed as*

$$y = \alpha y_1 + \beta y_2$$

for some α, $\beta \in \mathbb{R}$.

Proof. Let y be a solution of (2), and for $x_0 \in I$, let

$$y_0 := y(x_0), \quad z_0 := y'(x_0).$$

Let $W(x)$ be the Wronskian of y_1, y_2. Since y_1 and y_2 are linearly independent solutions of (2), by Theorem 2.5, $W(x_0) \neq 0$. Hence, there exists a unique pair $\alpha, \beta)$ of real numbers such that

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}.$$

Let

$$\varphi(x) = \alpha y_1(x) + \beta y_2(x), \quad x \in I.$$

Then φ is a solution of (2) satisfying

$$\varphi(x_0) = \alpha y_1(x_0) + \beta y_2(x_0) = y_0, \quad \varphi'(x_0) = \alpha y_1'(x_0) + \beta y_2'(x_0) = z_0.$$

By the existence and uniqueness theorem, we obtain $\varphi(x) = y(x)$ for all $x \in I$, i.e.,

$$y = \alpha y_1 + \beta y_2.$$

□

Theorem 2.5 and Theorem 2.8 give Theorem 2.3.

Now, the question is how to get linearly independent solutions for (2).

THEOREM 2.9. *Let y_1 be a nonzero solution of (2). Then*

$$y_2(x) := y_1(x) \int \frac{\psi(x)}{y_1(x)^2} dx, \quad \psi(x) := e^{-\int_{x_0}^{x} a(t)dt},$$

is a solution of (2), and y_1, y_2 are linearly independent.

Proof. Let $y_2(x) = y_1(x)\varphi(x)$,where

$$\varphi(x) := \int \frac{\psi(x)}{y_1(x)^2} dx, \quad \psi(x) := e^{-\int_{x_0}^{x} a(t)dt}.$$

Then

$$y_2' = y_1\varphi' + y_1'\varphi, \quad y_2'' = y_1\varphi'' + y_1'\varphi' + y_1'\varphi' + y_1''\varphi = y_1\varphi'' + 2y_1'\varphi' + y_1''\varphi.$$

Hence,

$$\begin{aligned} y_2'' + ay' + by_2 &= y_1\varphi'' + 2y_1'\varphi' + y_1''\varphi + a(y_1\varphi' + y_1'\varphi) + by_1\varphi \\ &= y_1\varphi'' + 2y_1'\varphi' + (y_1'' + ay_1' + by_1\varphi)\varphi + ay_1\varphi' \\ &= y_1\varphi'' + 2y_1'\varphi' + ay_1\varphi' \end{aligned}$$

Note that

$$\varphi' = \frac{\psi(x)}{y_1(x)^2}, \quad \text{i.e., } y_1^2\varphi' = \psi.$$

Hence

$$y_1^2\varphi'' + 2y_1y_1'\varphi' = \psi' \quad \text{i.e., } y_1(y_1\varphi'' + 2y_1'\varphi') = \psi'$$

so that

$$y_2'' + ay' + by_2 = y_1\varphi'' + 2y_1'\varphi' + ay_1\varphi' = \frac{\psi'}{y_1} + \frac{a\psi}{y_1} = \frac{\psi' + a\psi}{y_1} = 0.$$

Clearly, y_1 and y_2 are linearly independent. □

Motivation for the above expression for y_2:

If y_1 and y_2 are solutions of (2), then we know that

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{y_1y_2' - y_2y_1'}{y_1^2} = \frac{W(x)}{y_1^2} = \frac{Ce^{-\int_{x_0}^{x} a(t)dt}}{y_1^2}.$$

Hence,

$$y_2 = y_1 \int \left(\frac{Ce^{-\int_{x_0}^{x} a(t)dt}}{y_1^2}\right) dx.$$

2.2 Second order linear homogeneous ODE with constant coefficients

The DE in this case is of the form

$$y'' + py' + qy = 0, \tag{1}$$

where p, q are real constants. Let us look for a solution (1) in the form $y = e^{\lambda x}$ for some λ, real or complex. Assuming that such a solution exists, from (1) we have

$$(\lambda^2 + p\lambda + q)e^{\lambda x} = 0$$

so that λ must satisfy the **auxiliary equation**:

$$\lambda^2 + p\lambda + q = 0. \tag{2}$$

We have the following cases:

1. (2) has two distinct real roots λ_1, λ_2,
2. (2) has two distinct complex roots $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$,
3. (2) has a multiple root λ.

- In case 1, $e^{\lambda_1 x}, e^{\lambda_2 x}$ are linearly independent solutions.
- In case 2, $e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x$ are linearly independent solutions.

- In case 1, $e^{\lambda x}, xe^{\lambda x}$ are linearly independent solutions.

Example 2.10.

$$y'' + y' - 2y = 0$$

Auxiliary equation: $\lambda^2 + \lambda - 2 = 0$ has two distinct real roots: $\lambda_1 = 1, \quad \lambda_2 = -2$.
General solution: $y = C_1 e^x + C_2 e^{-2x}$. ♢

Example 2.11.

$$y'' + 2y' + 5y = 0$$

Auxiliary equation: $\lambda^2 + 2\lambda + 5 = 0$ has two complex roots: $-1 + i2, \quad = -1 - i2$.
General solution: $y = e^{-x}[C_1 \cos 2x + C_2 \sin 2x]$. ♢

Example 2.12.

$$y'' - 4y' + 4y = 0$$

Auxiliary equation: $\lambda^2 - 4\lambda + 4 = 0$ has a multiple root: $\lambda_0 = 2$.
General solution: $y = e^{2x}[C_1 + C_2 e^{2x}]$. ♢

2.3 Second order linear non-homogeneous ODE

Consider the nonhomogeneous ODE:

$$y'' + a(x)y' + b(x)y = f(x), \tag{1}$$

We observe that if y_0 is a solution of the homogeneous equation

$$y'' + a(x)y' + b(x)y = 0 \tag{2}$$

and y^* is a particular solution of the nonhomogeneous equation (1), then

$$y = y_0 + y^*$$

is a solution of the nonhomogeneous equation (1). Also, if y^* is a particular solution of the nonhomogeneous equation (1) and if y is any solution of the nonhomogeneous equation (1), then $y - y^*$ is a solution of the homogeneous equation (2). Thus, knowing a particular solution y^* of the nonhomogeneous equation (1) and a general solution $\bar{y}$ of homogeneous equation (2), we obtain a general solution of the nonhomogeneous equation (1) as

$$y = \bar{y} + y^*.$$

If the coefficients are constants, then we know a method of obtaining two linearly independent solutions for the homogeneous equation (2), and thus we obtain a general solution for the homogeneous equation (2).

How to get a particular solution for the nonhomogeneous equation (1)?

2.3.1 Method of variation of parameters

Suppose y_1 and y_2 are linearly independent solutions of the homogeneous ode:

$$y'' + a(x)y' + b(x)y = 0. \tag{2}$$

The, look for a solution of (1) in the form

$$y = u_1y_1 + u_2y_2$$

where u_1 and u_2 are unctions to be determined. Assume for a moment that such a solution exists. Then

$$y' = u_1y_1' + u_2y_2' + u_1'y_1 + u_2'y_2.$$

We shall look for u_1, u_2 such that

$$u_1'y_1 + u_2'y_2 = 0 \qquad (3).$$

Then, we have

$$y' = u_1y_1' + u_2y_2', \tag{4}$$

$$y'' = u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2'. \tag{5}$$

Substituting (4-5) in (1),

$$(u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2') + a(x)(u_1y_1' + u_2y_2') + b(x)(u_1y_1 + u_2y_2) = f(x),$$

i.e.,

$$u_1[y_1'' + a(x)y_1'b(x)y_1] + u_2[y_2'' + a(x)y_2'b(x)y_2] + u_1'y_1' + u_2'y_2' = f(x),$$

i.e.,

$$u_1'y_1' + u_2'y_2' = f(x). \tag{6}$$

Now, (3) and (6):

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

gives

$$u_1' = -\frac{y_2f}{W}, \quad u_2' = \frac{y_1f}{W}.$$

Hence,

$$u_1 = -\int \frac{y_2f}{W} + C_1, \quad u_2 = \int \frac{y_1f}{W} + C_2.$$

Thus,

$$y = \left(-\int \frac{y_2f}{W} + C_1\right) y_1 + \left(\int \frac{y_1f}{W} + C_2\right) y_2$$

is the general solution. Thus we have proved the following theorem.

THEOREM 2.13. *If y_1, y_2 are linearly independent solutions of the homogeneous equation (2), and if $W(x)$ is their Wronskian, then a general solution of the nonhomogeneous equation (1) is given by*

$$y = u_1 y_1 + u_2 y_2,$$

where

$$u_1 = -\int \frac{y_2 f}{W} + C_1, \quad u_2 = \int \frac{y_1 f}{W} + C_2.$$

Analogously, it the following theorem also can be proved:

THEOREM 2.14. *If y_1, $y_2, \ldots, y_n$ are linearly independent solutions of the homogeneous equation*

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y^{(1)} + a_n(x)y = 0,$$

where $a_1, a_2, \ldots, a_n$ are continuous functions on an interval I, and if $W(x)$ is their Wronskian, i.e.,

$$W(x) = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix},$$

then a general solution of the nonhomogeneous equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y^{(1)} + a_n(x)y = f(x)$$

is given by

$$y = (u_1 + C_1)y_1 + (u_2 + C_2)y_2 + \cdots + (u_n + C_n)y_n,$$

where $u_1', u_2', \ldots, u_n'$ are obtained by solving the system

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \cdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ f \end{bmatrix}.$$

Remark 2.15. Suppose the right hand side of (1) is of the form $f(x) = f_1(x) + f_2(x)$. Then it can be easily seen that:

If y_1 and y_2 are solutions of

$$y'' + a(x)y' + b(x)y = f_1(x), \qquad y'' + a(x)y' + b(x)y = f_2(x),$$

respectively, then $y_1 + y_2$ are solutions of

$$y'' + a(x)y' + b(x)y = f_1(x) + f_2(x).$$

$\diamondsuit$

2.3.2 Method of undetermined coefficients

This method is when the coefficients of (1) are constants and f is of certain special forms. So, consider

$$y'' + py' + qy = f, \tag{1}$$

where p, q are constants.

Case (i): $f(x) = P(x)e^{\alpha x}$, where P is a polynomial of degree n, and $\alpha \in \mathbb{R}$:

We look for a solution of the form

$$y = Q(x)e^{\alpha x},$$

where Q is a polynomial of degree n Substituting the above expression in the DE, we obtain:

$$[Q'' + (2\alpha + p)Q' + (\alpha^2 + p\alpha + q)Q]e^{\alpha x} = P(x)e^{\alpha x}.$$

Thus, we must have

$$Q'' + (2\alpha + p)Q' + (\alpha^2 + p\alpha + q)Q = P(x).$$

Note that, the above equation is an identity only if $\alpha^2 + p\alpha + q \neq 0$, i.e., α is not a root of the auxiliary equation $\lambda^2 + p\lambda + q = 0$. In such case, we can determine Q by comparing coefficients of powers of x^k for $k = 0, 1, \ldots, n$.

If α is a root of the auxiliary equation $\lambda^2 + p\lambda + q = 0$, then we must look for a solution of the form

$$y = \widetilde{Q}(x)e^{\alpha x},$$

where $\widetilde{Q}$ is a polynomial of degree $n + 1$, or we must look for a solution of the form

$$y = xQ(x)e^{\alpha x},$$

where Q is a polynomial of degree n. Proceeding as above we can determine Q provided $2\alpha + p \neq 0$, i.e., if α is not a double root of the auxiliary equation $\lambda^2 + p\lambda + q = 0$.

If α is a double root of the auxiliary equation $\lambda^2 + p\lambda + q = 0$, then we must look for a solution of the form

$$y = \widehat{Q}(x)e^{\alpha x},$$

where $\widehat{Q}$ is a polynomial of degree $n + 2$, or we must look for a solution of the form

$$y = x^2Q(x)e^{\alpha x},$$

where Q is a polynomial of degree n, which we can determine by comparing coefficients of powers of x.

Case (ii): $f(x) = P_1(x)e^{\alpha x}\cos\beta x + P_1(x)e^{\alpha x}\sin\beta x$, where P_1 and P_2 are polynomials and α, β are real numbers:

We look for a solution of the form

$$y = Q_1(x)e^{\alpha x}\cos\beta x + Q_1(x)e^{\alpha x}\sin\beta x,$$

where Q_1 and Q_2 are polynomials with

$$\deg Q_j(x) = \max\{P_1(x),\, P_2(x)\}, \quad j \in \{1, 2\}.$$

Substituting the above expression in the DE, *we obtain the coefficients of* Q_1, Q_2 *if* $\alpha + i\beta$ *is not a root of the auxiliary equation* $\lambda^2 + p\lambda + q = 0$.

If $\alpha + i\beta$ is a simple root of the auxiliary equation $\lambda^2 + p\lambda + q = 0$, then we look for a solution of the form

$$y = x[Q_1(x)e^{\alpha x}\cos\beta x + Q_1(x)e^{\alpha x}\sin\beta x],$$

where Q_1 and Q_2 are polynomials with $\deg Q_j(x) = \max\{P_1(x),\, P_2(x)\}, j \in \{1, 2\}$.

The following example illustrates the second part of case (ii) above:

Example 2.16. [2] We find the general solution of

$$y'' + 4y = x\,\sin 2x.$$

The auxiliary equation corresponding to the homogeneous equation $y'' + 4y = 0$ is:

$$\lambda^2 + 4 = 0.$$

Its solutions are $\lambda = \pm 2i$. Hence, the general solution of the homogenous equation is:

$$y_0 = A\cos 2x + B\sin 2x.$$

Note that the non-homogenous term, $f(x) = x\sin 2x$, is of the form

$$f(x) = P_1(x)e^{\alpha x}\cos\beta x + P_1(x)e^{\alpha x}\sin\beta x,$$

with $P_1(x) = 0, \quad \alpha = 0, \beta = 2$. Also, $2i = \alpha + i\beta$ is a simple root of the auxiliary equation. Hence, a particular solution is of the form

$$y = x[Q_1(x)e^{\alpha x}\cos\beta x + Q_1(x)e^{\alpha x}\sin\beta x],$$

where Q_1 and Q_2 are polynomials with $\deg Q_j(x) = \max\{P_1(x),\, P_2(x)\} = 1$. Thus, a a particular solution is of the form

$$y = x[(A_0 + A_1 x)\cos 2x + (B_0 + B_1 x)\sin 2x].$$

Differentiating:

$$y' = [A_0 + (2A_1 + 2B_0)x + 2B_1x^2]\cos 2x + [B_0 + (2B_1 - 2A_0)x - 2A_1x^2]\sin 2x,$$

[2]This example is included in the notes on November 23, 2012 – mtnair.

$$\begin{aligned}
y'' + 4y &= 2[B_0 + (2B_1 - 2A_0)x - 2A_1x^2]\cos 2x \\
&\quad -2[A_0 + (2A_1 + 2B_0)x + 2B_1x^2]\sin 2x \\
&\quad +[(2B_1 - 2A_0) - 4A_1x]\sin 2x + [(2A_1 + 2B_0) + 4B_1x]\cos 2x \\
&\quad +4x[(A_0 + A_1x)\cos 2x + (B_0 + B_1x)\sin 2x].
\end{aligned}$$

Hence, $y'' + 4y = x\sin 2x$ if and only if

$$2[B_0 + (2B_1 - 2A_0)x - 2A_1x^2] + [(2A_1 + 2B_0) + 4B_1x] + 4x(A_0 + A_1x) = 0,$$

$$-2[A_0 + (2A_1 + 2B_0)x + 2B_1x^2] + [(2B_1 - 2A_0) - 4A_1x] + 4x(B_0 + B_1x) = x$$

$\iff$

$$A_0 = 0, \quad A_1 = -\frac{1}{8}, \quad B_0 = \frac{1}{16}, \quad B_1 = 0,$$

so that

$$y = x[(A_0 + A_1x)\cos 2x + (B_0 + B_1x)\sin 2x] = -\frac{x^2}{8}\cos 2x + \frac{x}{16}\sin 2x.$$

Thus, the general solution of the equation is:

$$A\cos 2x + B\sin 2x - \frac{x^2}{8}\cos 2x + \frac{x}{16}\sin 2x.$$

$\diamondsuit$

Remark 2.17. The above method can be generalized, in a *natural way*, to higher order equation

$$y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y^{(1)} + a_ny = f(x)$$

where f is of the form

$$f(x) = P_1(x)e^{\alpha x}\cos\beta x + P_1(x)e^{\alpha x}\sin\beta x$$

with P_1 and P_2 being polynomials and α, β are real numbers. $\diamondsuit$

2.3.3 Equations reducible to constant coefficients case

A particular type of equations with non-constant coefficients can be reduced to the ones with constant coefficients. here it is: Consider

$$x^ny^{(n)} + a_1x^{n-1}y^{(n-1)} + \cdots + a_{n-1}xy^{(1)} + a_ny = f(x). \tag{1}$$

In this case, we take the change of variable: $x \mapsto z$ defined by

$$x = e^z.$$

Then the equation (1) can be brought to the form

$$D^ny + b_1D^{n-1}y + \cdots + b_{n-1}Dy + a_ny = f(e^z), \quad D := \frac{d}{dz},$$

where $b_1, b_2, \ldots, b_n$ are constants. Let us consider the case of $n = 2$:

$$x^2 y'' + a_1 x y' + a_2 y = f(x).$$

Taking $x = e^z$,

$$\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = y'x,$$

$$\frac{d^2y}{dz^2} = \frac{d}{dz}(y'x) = \frac{dy'}{dz}x + y'\frac{dx}{dz} = y''x^2 + y'x = y''x^2 + \frac{dy}{dz}.$$

Hence we have

$$x^2 y'' + a_1 x y' + a_2 y = \left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + a_1 \frac{dy}{dz} + a_2 y = \frac{d^2y}{dz^2} + (a_1 - 1)\frac{dy}{dz} + a_2 y.$$

Thus, the equation takes the form:

$$\frac{d^2y}{dz^2} + (a_1 - 1)\frac{dy}{dz} + a_2 y = f(e^z).$$

Note also that

$$\begin{aligned}
\frac{d^3y}{dz^3} &= \frac{d}{dz}(y''x^2 + y'x) = \frac{dy''}{dz}x^2 + y''2x\frac{dx}{dz} + y''x^2 + y'x \\
&= y'''x^3 + 2y''x^2 + y''x^2 + y'x \\
&= y'''x^3 + 3\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + \frac{dy}{dz}.
\end{aligned}$$

Hence,

$$\begin{aligned}
x^3 y''' + a x^2 y'' + b x y' + c y &= \frac{d^3y}{dz^3} - 3\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) - \frac{dy}{dz} + a\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + b\frac{dy}{dz} + cy \\
&= \frac{d^3y}{dz^3} + (a-3)\frac{d^2y}{dz^2} + (b - a + 3)\frac{dy}{dz} + cy.
\end{aligned}$$

3 System of first order linear homogeneous ODE

Consider the system:

$$\begin{aligned}\frac{dx_1}{dt} &= ax_1 + bx_2\\ \frac{dx_2}{dt} &= cx_1 + dx_2\end{aligned}$$

The above system can be written in matrix notation as:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{1}$$

or more compactly as:

$$\frac{dX}{dt} = AX,$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Here, we used the convention:

$$\frac{d}{dt}\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} f' \\ g' \end{bmatrix}.$$

In this case we look for a solution of the form

$$X = \begin{bmatrix} \alpha_1 e^{\lambda t} \\ \alpha_2 e^{\lambda t} \end{bmatrix} =: \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} e^{\lambda t}.$$

Substituting this into the system of equations we get

$$\lambda \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} e^{\lambda t} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} e^{\lambda t}.$$

Equivalently,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \lambda \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}.$$

That is,

$$\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2}$$

Thus, if λ_0 is a root of the equation

$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0, \tag{2}$$

then there is a nonzero vector $[\alpha_1, \alpha_2]^T$ satisfying (2), and $X = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} e^{\lambda_0 t}$ is a solution of the system (1).

Definition 3.1. The equation (3) is called the **auxiliary equation** for the system (1). $\diamondsuit$

Let us consider the following cases:

Case (i): Suppose the roots of the auxiliary equation (3) are real distinct, say λ_1 and λ_2. Suppose

$$\begin{bmatrix} \alpha_1^{(1)} \\ \alpha_2^{(1)} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \alpha_1^{(2)} \\ \alpha_2^{(2)} \end{bmatrix}$$

are nonzero solutions of (2) corresponding to $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively. Then, the vector valued functions

$$X_1 = \begin{bmatrix} \alpha_1^{(1)} \\ \alpha_2^{(1)} \end{bmatrix} e^{\lambda_1 t}, \quad X_2 = \begin{bmatrix} \alpha_1^{(2)} \\ \alpha_2^{(2)} \end{bmatrix} e^{\lambda_2 t}$$

are solutions of (1), and they are linearly independent. In this case, the general solution of (1) is given by $C_1 X_1 + C_2 X_2$.

Case (ii): Suppose the roots of the auxiliary equation (3) are complex non-real. Since the entries of the matrix are real, these roots are conjugate to each other. Thus, they are of the form $\alpha + i\beta$ and $\alpha - i\beta$ for $\beta \neq 0$. Suppose $\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ be a nonzero solution of (2) corresponding to $\lambda = \alpha + i\beta$. The numbers α_1 and α_2 need not be real. Thus,

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1^{(1)} + i\alpha_1^{(2)} \\ \alpha_2^{(1)} + i\alpha_2^{(2)} \end{bmatrix}.$$

Then, the vector valued function

$$X := \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} e^{(\alpha+i\beta)t} = \begin{bmatrix} \alpha_1^{(1)} + i\alpha_1^{(2)} \\ \alpha_2^{(1)} + i\alpha_2^{(2)} \end{bmatrix} e^{\alpha t}[\cos\beta t + i\sin\beta t]$$

is a solution of (1). Note that

$$X = X_1 + iX_2,$$

where

$$X_1 = \begin{bmatrix} \alpha_1^{(1)}\cos\beta t - \alpha_1^{(2)}\sin\beta t \\ \alpha_2^{(1)}\cos\beta t - \alpha_2^{(2)}\sin\beta t \end{bmatrix} e^{\alpha t}, \quad X_2 = \begin{bmatrix} \alpha_1^{(1)}\sin\beta t + \alpha_1^{(2)}\cos\beta t \\ \alpha_2^{(1)}\sin\beta t + \alpha_2^{(2)}\cos\beta t \end{bmatrix} e^{\alpha t}.$$

We see that X_1 and X_2 are also are solutions of (1), and they are linearly independent. In this case, a general solution of (1) is given by $C_1 X_1 + C_2 X_2$.

Case (iii): Suppose λ_0 is a double root of the auxiliary equation (3). In this case there are two subcases:

- There are linearly independent solutions for (2).
- There is only one (up to scalar multiples) nonzero solution for (2).

In the first case if

$$\begin{bmatrix} \alpha_1^{(1)} \\ \alpha_2^{(1)} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \alpha_1^{(2)} \\ \alpha_2^{(2)} \end{bmatrix}$$

are the linearly independent solutions of (2) corresponding to $\lambda = \lambda_0$, then the vector valued functions

$$X_1 = \begin{bmatrix} \alpha_1^{(1)} \\ \alpha_2^{(1)} \end{bmatrix} e^{\lambda_0 t}, \quad X_2 = \begin{bmatrix} \alpha_1^{(2)} \\ \alpha_2^{(2)} \end{bmatrix} e^{\lambda_0 t}$$

are solutions of (1), and the general solution of (1) is given by

$$C_1 X_1 + C_2 X_2.$$

In the second case, let $\underline{u} := \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ is a nonzero solution of (2) corresponding to $\lambda = \lambda_0$, and let $\underline{v} := \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$ is such that

$$(A - \lambda_0 I)\underline{v} = \underline{u}.$$

Then

$$X = C_1 \underline{u} e^{\lambda_0 t} + C_2 [t\underline{u} + \underline{v}] e^{\lambda_0 t}$$

is the general solution.

Remark 3.2. Another method of solving a system is to convert the given system into a second order system for one of x_1 and x_2, and obtain the other. $\diamondsuit$

4 Power series method

4.1 The method and some examples

Consider the differential equation:

$$y'' + f(x)y' + g(x)y = r(x). \tag{1}$$

We would like to see if the above equation has a solution of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n \tag{2}$$

in some interval I containing some known x_0, where $c_0, c_1, \ldots$ are to determined.

Recall from calculus: Suppose the power series $\displaystyle\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges at some point other than x_0.

- There exists $\rho > 0$ such that the series converges at every x with $|x - x_0| < \rho$.
- The series diverges at every x with $|x - x_0| > \rho$.
- $\displaystyle\sum_{n=0}^{\infty} a_n (x - x_0)^n = 0$ implies $a_n = 0$ for all $n = 0, 1, 2, \ldots$.
- The series can be differentiated term by term in the interval $(x_0 - r, x_0 + \rho)$ any number of times, i.e.,
$$\frac{d^k}{dx^k} \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (x - x_0)^{n-k}$$
for every x with $|x - x_0| < \rho$ and for every $k \in \mathbb{N}$.
- If $f(x) := \sum_{n=0}^{\infty} a_n (x - x_0)^n$ for $|x - x_0| < \rho$, then $a_n = \dfrac{f^{(n)}(x_0)}{n!}$.

The above number ρ is called the radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty} a_n (x - x_0)^n$.

Definition 4.1. A (real valued) function f defined in a neighbourhood of a point $x_0 \in \mathbb{R}$ is said to be **analytic** at x_0 if it can be expressed as

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad |x - x_0| < \rho,$$

for some $\rho > 0$, where $a_0, a_1, \ldots$ are real numbers. $\diamondsuit$

Recall that if $p(x)$ and $q(x)$ are polynomials given by

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad q(x) = b_0 + b_1 x + \cdots + b_n x^n,$$

then

$$p(x)q(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + \cdots + (a_0b_n + a_1b_{n-1} + \cdots + a_nb_0)x^n.$$

Motivated by this, for convergent power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ and $\sum_{n=0}^{\infty} b_n(x-x_0)^n$, we define

$$\Big(\sum_{n=0}^{\infty} a_n(x-x_0)^n\Big)\Big(\sum_{n=0}^{\infty} b_n(x-x_0)^n\Big) = \sum_{n=0}^{\infty} c_n(x-x_0)^n, \quad c_n := \sum_{k=0}^{n} a_kb_{n-k}.$$

Now, it may be too much to expect to have a solution of the form (2) for a differential equation (1) for arbitrary continuous functions f, $g\,r$. Note that we require the solution to have only second derivative, whereas we are looking for a solution having a series expansion; in particular, differentiable infinitely many times. But, it may not be too much expect to have a solution of the form (2) if f, g, r also have power series expansions about x_0. **Power series method** is based on such assumptions.

The idea is to consider those cases when f, g, r also have power series expansions about x_0, say

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n, \quad g(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^n, \quad r(x) = \sum_{n=0}^{\infty} d_n(x-x_0)^n..$$

Then substitute the expressions for f, g, r, y and obtain the coefficients $c_n, n \in \mathbb{N}$, by comparing coefficients of $(x-x_0)^k$ for $k = 0, 1, 2, \ldots$.

Note that this case includes the situation when:

- Any of the functions f, g, r is a polynomial,
- Any of the functions f, g, r is a rational function, i.e., function of the form $p(x)/q(x)$ where $p(x)$ and $q(x)$ are polynomials, and in that case the point x_0 should not be a zero of $q(x)$.

Example 4.2.

$$y'' + y = 0. \tag{$*$}$$

In this case, $f = 0$, $g = 0$, $r = 0$. So, we may assume that the equation has a solution power series expansion around any point $x_0 \in \mathbb{R}$. For simplicity, let $x_0 = 0$, and assume that the solution is of the form $y = \sum_{n=0}^{\infty} c_nx^n$. Note that

$$(*) \iff \sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} + \sum_{n=0}^{\infty} c_nx^n = 0 \iff \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + \sum_{n=0}^{\infty} c_nx^n = 0$$

$$\iff \sum_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} + c_n]x^n = 0 \iff (n+2)(n+1)c_{n+2} + c_n \quad \forall n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$$

$$\iff (n+2)(n+1)c_{n+2} = -\frac{c_n}{(n+2)(n+1)} \quad \forall n \in \mathbb{N}_0$$

$$\iff c_{2n} = \frac{(-1)^n a_0}{(2n)!}, \quad c_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!} \quad \forall n \in \mathbb{N}_0.$$

Thus, if $y = \sum_{n=0}^{\infty} c_n x^n$ is a solution of $(*)$, then

$$y = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} + \sum_{n=0}^{\infty} c_{2n+1} x^{2n+1} = c_0 \cos x + c_1 \sin x$$

for arbitrary c_0 and c_1. We can see that this, indeed, is a solution. $\diamondsuit$

The following theorem specifies conditions under which a power series solution is possible.

THEOREM 4.3. *Let $p, q\, r$ be analytic at a point x_0. Then every solution of the equation*

$$y'' + p(x)y' + q(x)y = r(x)$$

can be represented as a power series in powers of $x - x_0$.

4.2 Legendre's equation and Legendre polynomials

The differential equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0 \qquad (*)$$

is called *Legendre equation.* Here, α is a real constant. Note that the above equation can also be written as

$$\frac{d}{dx}\left[(1 - x^2)\frac{dy}{dx}\right] + \alpha(\alpha + 1)y = 0.$$

Note that $(*)$ can also be written as

$$y'' - \frac{2xy'}{1 - x^2} + \frac{\alpha(\alpha + 1)y}{1 - x^2} = 0.$$

It is of the form (1) with

$$f(x) = -\frac{2x}{1 - x^2}, \quad g(x) = \frac{\alpha(\alpha + 1)}{1 - x^2}, \quad r(x) = 0.$$

Clearly, f, g, r are rational functions, and have power series expansions around the point $x_0 = 0$. Let us assume that a solution of $(*)$ is of the form $y = \sum_{n=0}^{\infty} c_n x^n$. Substituting the expressions for y, y', y'' into $(*)$, we obtain

$$(1 - x^2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x\sum_{n=1}^{\infty} nc_n x^{n-1} + \alpha(\alpha + 1)\sum_{n=0}^{\infty} c_n x^n = 0,$$

i.e.,

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=1}^{\infty} 2nc_n x^n + \alpha(\alpha + 1)\sum_{n=0}^{\infty} c_n x^n = 0,$$

i.e.,

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=1}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} \alpha(\alpha + 1)c_n x^n = 0.$$

Equating coefficients of x^k to 0 for $k \in \mathbb{N}_0$, we obtain

$$2c_2 + \alpha(\alpha+1)c_0 = 0, \quad 6c_3 - 2c_1 + \alpha(\alpha+1)c_1 = 0,$$

$$(n+2)(n+1)c_{n+2} + [-n(n-1) - 2n + \alpha(\alpha+1)]c_n = 0,$$

i.e.,

$$2c_2 + \alpha(\alpha+1)c_0 = 0, \quad 6c_3 + [-2 + \alpha(\alpha+1)]c_1 = 0,$$

$$(n+2)(n+1)c_{n+2} + (\alpha - n)(\alpha + n + 1)c_n = 0,$$

i.e.,

$$c_2 = -\frac{\alpha(\alpha+1)}{2}c_0, \quad c_3 = \frac{-2+\alpha(\alpha+1)}{6}c_1, \quad c_{n+2} = -\frac{(\alpha-n)(\alpha+n+1)}{(n+2)(n+1)}c_n.$$

Note that if $\alpha = k$ is a positive integer, then coefficients of x^{n+2} is zero for $n \in \{k, k+1, \ldots\}$. Thus, in this case we have $y = y_1(x) + y_2(x)$, where:

- If $\alpha = k$ is an even integer, then $y_1(x)$ is a polynomial of degree k with only even powers of x, and $y_2(x)$ is a power series with only odd powers of x,
- If $\alpha = k$ is an odd integer, then $y_2(x)$ is a polynomial of degree k with only odd powers of x, and $y_1(x)$ is a power series with only even powers of x.

Now, suppose $\alpha = k$ is a positive integer. Then, from the iterative formula

$$c_{n+2} = -\frac{(\alpha-n)(\alpha+n+1)}{(n+2)(n+1)}c_n$$

we have $c_k \neq 0$ and $c_{k+2} = 0$ so that

$$c_{k+2j} = 0 \quad \text{for} \quad j \in \mathbb{N}.$$

Thus,

$$c_{k-2} = -\frac{k(k-1)}{2(2k-1)}c_k,$$

$$c_{k-4} = -\frac{(k-2)(k-3)}{4(2k-3)}c_{k-2} = (-1)^2\frac{k(k-1)(k-2)(k-3)}{2.4.(2k-1)(2k-3)}c_k.$$

$$c_{k-6} = -\frac{(k-4)(k-5)}{6(2k-5)}c_{k-4} = (-1)^3\frac{k(k-1)(k-2)(k-3)(k-4)(k-5)}{2.4.6(2k-1)(2k-3)(2k-5)}c_k.$$

In general, for $2\ell < k$,

$$\begin{aligned}
c_{k-2\ell} &= (-1)^\ell \frac{k(k-1)(k-2)\cdots(k-2\ell+1)}{[2.4.\cdots(2\ell)](2k-1)(2k-3)\cdots(2k-2\ell+1)}c_k \\
&= (-1)^\ell \frac{k!(2k-2)(2k-4)\cdots(2k-2\ell)}{(k-2\ell)!2^\ell\ell!(2k-1)(2k-2)(2k-3)(2k-4)\cdots(2k-2\ell+1)(2k-2\ell)}c_k \\
&= (-1)^\ell \frac{k!2^\ell(k-1)(k-2)\cdots(k-\ell)}{(k-2\ell)!2^\ell\ell!(2k-1)(2k-2)(2k-3)(2k-4)\cdots(2k-2\ell+1)(2k-2\ell)}c_k \\
&= (-1)^\ell \frac{k!(k-1)!(2k-2\ell-1)!}{(k-2\ell)!\ell!(k-\ell-1)!(2k-1)!}c_k
\end{aligned}$$

Taking

$$c_k := \frac{(2k)!}{2^k(k!)^2}$$

it follows that

$$c_{k-2\ell} = (-1)^\ell \frac{(2k-2\ell)!}{2^k \ell!(k-\ell)!(k-2\ell)!}.$$

Definition 4.4. The polynomial

$$P_n(x) = \sum_{\ell=0}^{M_n} (-1)^\ell \frac{(2n-2\ell)!}{2^n \ell!(n-\ell)!(n-2\ell)!} x^{n-2\ell}$$

is called the **Legendre polynomial** of degree n. Here, $M_n = n/2$ if n is even and $M_n = (n-1)/2$ if n is odd. $\diamondsuit$

Recall

$$P_n(x) = \sum_{k=0}^{M_n} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}.$$

It can be seen that

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3}{2}(x^2-1), \quad P_2(x) = \frac{1}{5}(5x^3-3x),$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x).$$

$$P_n(-x) = \sum_{k=0}^{M_n} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} (-x)^{n-2k} = (-1)^n P_n(x).$$

Rodrigues' formula: $P_n(x) = \dfrac{1}{n!2^n} \dfrac{d^n}{dx^n}(x^2-1)^n.$

Let

$$f(x) = (x^2-1)^n = \sum_{r=0}^{n} (-1)^r ({}^nC_r) x^{2n-2r}.$$

Then

$$f'(x) = \sum_{r=0}^{M_1} (-1)^r ({}^nC_r)(2n-2r) x^{2n-2r-1},$$

$$f''(x) = \sum_{r=0}^{M_2} (-1)^r ({}^nC_r)(2n-2r)(2n-2r-1) x^{2n-2r-2},$$

$$\begin{aligned}
f^n(x) &= \sum_{r=0}^{M_n} (-1)^r ({}^nC_r)[(2n-2r)(2n-2r-1)\cdot(2n-2r-n+1)] x^{2n-2r-n}, \\
&= \sum_{r=0}^{M_n} (-1)^r ({}^nC_r)[(2n-2r)(2n-2r-1)\cdot(n-2r+1)] x^{n-2r}, \\
&= \sum_{r=0}^{M_n} (-1)^r \frac{n!}{r!(n-r)!} \frac{(2n-2r)!}{(n-2r)!} x^{n-2r} \\
&= n!2^n P_n(x),
\end{aligned}$$

Generating function: $\dfrac{1}{\sqrt{1-2xu+u^2}} = \sum_{n=0}^{\infty} P_n(x)u^n.$

For a fraction β, we use the expansion:

$$(1+\alpha)^\beta = 1 + \sum_{n=1}^{\infty} ({}^\beta C_n)\alpha^n, \quad ({}^\beta C_n) := \frac{1}{n!}[\beta(\beta-1)\cdots(\beta-n+1)].$$

Thus, for $\beta = -1/2$,

$$\begin{aligned}
({}^{-1/2}C_n) &= \frac{1}{n!}\Big[\Big(-\frac{1}{2}\Big)\Big(-\frac{1}{2}-1\Big)\Big(-\frac{1}{2}-2\Big)\cdots\Big(-\frac{1}{2}-n+1\Big)\Big] \\
&= (-1)^n\frac{1}{n!}\Big[\Big(\frac{1}{2}\Big)\Big(\frac{3}{2}\Big)\Big(\frac{5}{2}\Big)\cdots\Big(\frac{2n-1}{2}\Big)\Big] \\
&= (-1)^n\frac{1}{n!2^n}\Big[\frac{(2n)!}{2^n n!}\Big] \\
&= (-1)^n\frac{(2n)!}{2^{2n}(n!)^2}.
\end{aligned}$$

Thus,

$$(1-\alpha)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} a_n\alpha^n, \quad a_n := (-1)^n\frac{(2n)!}{2^{2n}(n!)^2}.$$

Also,

$$(2xu-u^2)^n = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(2xu)^{n-k}(-u^2)^k = \sum_{k=0}^{n}(-1)^k 2^{n-k}\frac{n!}{k!(n-k)!}x^{n-k}u^{n+k}.$$

Thus,

$$(2xu-u^2)^n = \sum_{k=0}^{n} b_{n,k}x^{n-k}u^{n+k}, \quad b_{n,k} = (-1)^k 2^{n-k}\frac{n!}{k!(n-k)!}.$$

Taking $\alpha = 2xu-u^2$, we have

$$\begin{aligned}
(1-2xu+u^2)^{-\frac{1}{2}} &= \sum_{n=0}^{\infty} a_n\Big[\sum_{k=0}^{n} b_{n,k}x^{n-k}u^{n+k}\Big] \\
&= a_0 + a_1b_{1,0}xu + (a_1b_{1,1} + a_2b_{2,0}x^2)u^2 \\
&\quad +(a_2b_{2,1}x + a_3b_{3,0}x^3)u^3 \\
&\quad +(a_2b_{2,2} + a_3b_{3,1}x^2 + a_4b_{4,4}x^4)u^4 + \cdots \\
&= f_0(x) + f_1(x)u + f_2(x)u^2 + \cdots,
\end{aligned}$$

where

$$f_n(x) = \sum_{k=0}^{M_n} a_{n-k}b_{n-k,k}x^{n-2k}.$$

Since

$$a_{n-k}b_{n-k,k} = \frac{[2(n-k)]!}{(2^{n-k})^2[(n-k)!]^2}(-1)^k\frac{(n-k)!}{k!(n-2k)!}2^{n-2k} = (-1)^k\frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!},$$

we have

$$f_n(x) = P_n(x).$$

Thus,

$$\frac{1}{\sqrt{1-2xu-u^2}} = \sum_{n=0}^{\infty} P_n(x)u^n.$$

Note that, taking $x = 1$,

$$\sum_{n=0}^{\infty} u^n = \frac{1}{1-u} = \sum_{n=0}^{\infty} P_n(1)u^n$$

so that $P_n(1) = 1$ for all n.

Recurrence formulae:

1. $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$.

2. $nP_n = xP_n'(x) - P_{n-1}'(x)$.

3. $(2n+1)P_{n+1}(x) = P_{n+1}'(x) - nP_{n-1}'(x)$.

4. $P_{n+1}'(x) = xP_{n-1}'(x) - nP_{n-1}(x)$.

5. $(1-x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)]$.

Proofs. 1. Recall that the generating function for (P_n) is $(1-2xt+t^2)^{-\frac{1}{2}}$, i.e.,

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

Differentiating with respect to t:

$$(x-t)(1-2xt+t^2)^{-\frac{3}{2}} = \sum_{n=1}^{\infty} nP_n(x)t^{n-1}$$

$\iff$

$$(x-t)(1-2xt+t^2)^{-\frac{1}{2}} = (1-2xt+t^2)\sum_{n=1}^{\infty} nP_n(x)t^{n-1}$$

$\iff$

$$(x-t)\sum_{n=0}^{\infty} P_n(x)t^n = (1-2xt+t^2)\sum_{n=1}^{\infty} nP_n(x)t^{n-1} = (1-2xt+t^2)\sum_{n=0}^{\infty}(n+1)P_{n+1}(x)t^n.$$

Equating the coefficients of t^n, we obtain

$$xP_nx - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2x\,nP_n(x) + (n-1)P_{n-1}(x),$$

i.e.,

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

2. Differentiating with respect to t:

$$(x-t)(1-2xt+t^2)^{-\frac{3}{2}} = \sum_{n=1}^{\infty} nP_n(x)t^{n-1}$$

Differentiating with respect to x:

$$t(1-2xt+t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} P_n'(x)t^n$$

Hence,

$$(x-t)t(1-2xt+t^2)^{-\frac{3}{2}} = \sum_{n=1}^{\infty} nP_n(x)t^n = \sum_{n=0}^{\infty} nP_n(x)t^n$$

Thus,

$$(x-t)\sum_{n=0}^{\infty} P_n'(x)t^n = \sum_{n=0}^{\infty} nP_n(x)t^n$$

Equating the coefficients of t^n, we obtain $nP_n = xP_n'(x) - P_{n-1}'(x)$.

3. Differentiating the recurrence relation in (1) with respect to x and then using the expression for $xP_n'(x)$ from (2), we get the result in (3).

4. Differentiating the recurrence relation in (1) with respect to x leads to

$$(n+1)P_{n+1}'(x) = (2n+1)P_n(x) + (n+1)xP_n'(x) + n[xP_n'(x) - P_{n-1}'(x)].$$

Now, using (2) and replacing n by $n-1$ leads to the required relation.

5. Recurrence relation in (2) and (4) imply the required relation. □

Exercise 4.5. 1. Show that $P_n'(1) = \frac{n(n+1)}{2}$.
(Hint: Use the fact that $P_n(x)$ satisfies the Legendre equation.)

2. Using generating function derive
   (a) $P_n(-1) = (-1)^n$,
   (b) $P_n(-x) = (-1)^n P_n(x)$. (Hind: Replace x by $y := -x$ and then t by $\tau := -t$.)

3. Find values of $\int_{-1}^{1} x[P_n(x)]^2 dx$, $\int_{-1}^{1} x^2[P_n(x)]^2 dx$, $\int_{-1}^{1} x^2 P_{n+1}(x)P_{n-1}(x)dx$.
(Hint: Use recurrence formula.)

4. Prove that for every polynomial $q(x)$ of degree n, there exists a unique $(n+1)$-tuple $(a_0, a_1, \ldots, a_n)$ of real numbers such that $q(x) = a_0P_0(x) + a_1P_1(x) + \ldots a_nP_n(x)$.
(Hint: use induction on degree.)

♢

4.3 Power series solution around singular points

Look at the DE:

$$x^2y'' - (1+x)y = 0.$$

Does it have a nonzero solution of the form $\sum_{n=0}^{\infty} a_n x^n$? Following our method of substitution and determination of coefficients, it can be see that $a_n = 0$ for all $n \in \mathbb{N}_0$.

What went wrong?

Note that the above DE is same as

$$y'' - \frac{1+x}{x^2}y = 0,$$

which is of the form

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

with $p(x) = 0$ and $q(x) = \dfrac{1+x}{x^2}$. Note that $p(x)$ is not *analytic* at $x_0 = 0$.

Definition 4.6. A point $x_0 \in \mathbb{R}$ is called a **regular point** of (1) if $p(x)$ and $q(x)$ are analytic at x_0. If x_0 is not a regular point of (1), then it is called a **singular point** of (1). $\diamondsuit$

Example 4.7.

1. Consider $(x-1)y'' + xy' + \dfrac{y}{x} = 0$. This takes the form (1) with

   $$p(x) = \frac{x}{x-1}, \quad q(x) = \frac{1}{x(x-1)}.$$

   Note that $x = 0$ and $x = 1$ are singular points of the DE. All other points in $\mathbb{R}$ are regular points.

2. Consider the Cauchy equation: $x^2y'' + 2xy' - 2y = 0$. This takes the form (1) with

   $$p(x) = \frac{2}{x}, \quad q(x) = \frac{2}{x^2}.$$

   Note that $x = 0$ is the only singular point of this DE.

$\diamondsuit$

Definition 4.8. A singular point $x_0 \in \mathbb{R}$ of the DE (1) is called a **regular singular point** if $(x-x_0)p(x)$ and $(x-x_0)^2q(x)$ are analytic at x_0. Otherwise, x_0 is called an **irregular singular point** of (1). $\diamondsuit$

Example 4.9. Consider $x^2(x-2)y'' + 2y' + (x+1)y = 0$. This takes the form (1) with

$$p(x) = \frac{2}{x^2(x-2)}, \quad q(x) = \frac{x+1}{x^2(x-2)}.$$

Note that

$$xp(x) = \frac{2}{x(x-2)}, \quad x^2q(x) = \frac{x+1}{x-2},$$

$$(x-2)p(x) = \frac{2}{x^2}, \quad (x-2)^2q(x) = \frac{(x+1)(x-2)}{x^2}.$$

We see that

- $x = 0$ is an irregular singular point,
- $x = 2$ is a regular singular point.

$\diamondsuit$

Example 4.10. Consider the DE

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0,$$

where $a(x)$ and $b(x)$ are analytic at 0. Note that the above equation is of the form (1) with $p(x) = \dfrac{b(x)}{x}$ and $q(x) = \frac{c(x)}{x^2}$. Thus, 0 is a regular singular point of the given DE. $\diamondsuit$

4.3.1 Frobenius method

It is known that a DE of the form

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0, \tag{1}$$

where $a(x)$ and $b(x)$ are analytic at 0 has a solution of the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n,$$

for some real or complex number r and for some real numbers $a_0, a_1, a_2, \dots$ with $a_0 \neq 0$.

Note that $(*)$ is same as

$$x^2 y'' + x b(x) y' + c(x) y = 0 \tag{2}$$

and it reduces to the Euler–Cauchy equation when $b(x)$ and $c(x)$ are constant functions.

Substituting the expression for y in (2) into (1), we get:

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x b(x) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + c(x) = 0.$$

That is,

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + b(x) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + c(x) = 0. \tag{3}$$

Let

$$b(x) = \sum_{n=0}^{\infty} b_n x^n, \quad c(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Comparing comparing coefficients of x^r, we get

$$[r(r-1) + b_0 r + c_0] a_0 = 0.$$

This quadratic equation is called the **indicial equation** of (1).

Let r_1, r_2 be the roots of the indicial equation. Then one of the solutions is

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n,$$

where $a_0, a_1, \ldots$ are obtained by comparing coefficients of x^{n+r}, $n = 0, 1, 2, \ldots$, in (3) for $r = r_1$. Another solution, linearly independent of y_1 is obtained using the *method of variation of parameter*.

Recall that, in the *method of variation of parameter*,

- the second solution y_2 is assumed to be of the form $y_2(x) = u(x)y_1(x)$,
- substituting the expressions for y_2, y_2', y_2'' in (2),
- use the fact that $y_1(x)$ satisfies (2),
- obtain a first order ODE for $u(x)$, and
- solve it to obtain an expression for $u(x)$.

We have seen that

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)}}{[y_1(x)]^2}\, dx, \quad p(x) := \frac{a(x)}{x}.$$

In case $y_1(x)$ is already in a simple form, then the above expression can be used. Otherwise, one may use the above mentioned steps to reach appropriated expression for $y_2(x)$ by making use of the series expression for $y_1(x)$.

By the above procedure we have the following (see Kreiszig):

Case 1: If r_1 and r_2 distinct and not differing by an integer, then y_2 is of form

$$y_2(x) = x^{r_1} \sum_{n=0}^{\infty} A_n x^n.$$

Case 2: If $r_1 = r_2 = r$, say, i.e., r is a double root, then y_2 is of the form

$$y_2(x) = y_1(x) \ln(x) + x^r \sum_{n=1}^{\infty} A_n x^n.$$

Case 3: If r_1 and r_2 differ by an integer and $r_2 > r_1$, then y_2 is of the form

$$y_2(x) = k y_1(x) \ln(x) + x^{r_2} \sum_{n=0}^{\infty} A_n x^n.$$

The method described above is called the **Frobenius method**[3].

[3] George Frobenius (1849–1917) was a German mathematician.

Example 4.11. Let us ind linearly independent solutions for the **Euler-Cauchy equation**:

$$x^2y'' + b_0xy' + c_0y = 0.$$

Note that this is of the form (2) with $b(x) = b_0$, $c(x) = c_0$, constants. Assuming a solution is of the form $y = x^r \sum_{n=0}^{\infty} a_n x^n$, we obtain

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + b_0 \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + c_0 = 0.$$

Now, equating the coefficient of x^r to 0, we get the indicial equation as $[r(r-1) + b_0 r + c_0]a_0 = 0$, $a_0 \neq 0$, so that

$$r^2 - (1-b_0)r + c_0 = 0.$$

For a root r and $n \in \mathbb{N}$,

$$[(n+r)(n+r-1) + (n+r)b_0]a_n = 0, \text{ i.e., } (n+r)[(n+r-1) + b_0]a_n = 0, \textit{ i.e.,}$$

$$[(n+r-1) + b_0]a_n = 0 \quad \forall\, n \in \mathbb{N}.$$

We can take $a_n = 0$ for all $n \in \mathbb{N}$. Thus, $y_1(x) = x^r$. The other solution is given by

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)}}{[y_1(x)]^2}\, dx, \quad p(x) := \frac{a(x)}{x}.$$

Thus,

$$y_2(x) = x^r \int \frac{e^{-\int p(x)}}{x^{2r}}\, dx, \quad p(x) := \frac{b_0}{x}, \text{ i.e., } y_2(x) = x^r \int \frac{1}{x^{2r+b_0}}\, dx.$$

If r is a double root, then $2r + b_0 = 1$ so that

$$y_2(x) = x^r \ln(x).$$

If r is not a double root, then

$$y_2(x) = x^r \int \frac{1}{x^{2r+b_0}}\, dx = \frac{1}{-(2r+b_0-1)x^{r+b_0-1}}.$$

If $r = r_1$ and r_2 are the roots, then we have $r_1 + r_2 = 1 - b_0$ so that $r + b_0 - 1$ and hence,

$$y_2(x) = \frac{x^{r_2}}{-(2r_1 + b_0 - 1)}.$$

Thus, x^{r_1} and x^{r_2} are linearly independent solutions. $\diamondsuit$

Example 4.12. Consider the DE:

$$x(x-1)y'' + (3x-1)y' + y = 0. \tag{$*$}$$

This is of the form (1) with $b(x) = \dfrac{3x-1}{x-1}$, $c(x) = \dfrac{x}{x-1}$. Now, taking $y = x^r \sum_{n=0}^{\infty} a_n x^n$, we obtain from (1):

$$\begin{aligned}
x(x-1)y'' &= (x^2-x)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} \\
&= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} \\
(3x-1)y' &= (3x-1)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \\
&= \sum_{n=0}^{\infty}3(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}.
\end{aligned}$$

Hence, $(*)$:

$$\sum_{n=0}^{\infty}[(n+r)(n+r-1)+3(n+r)+1]a_n x^{n+r} + \sum_{n=0}^{\infty}[-(n+r)(n+r-1)-(n+r)]a_n x^{n+r-1} = 0.$$

Equating coefficient of x^{r-1} to 0, we get the indicial equation as $-r(r-1)-r=0$, i.e., $r^2=0$. Thus, $r=0$ is a double root of the indicial equation. Hence, we obtain:

$$\sum_{n=0}^{\infty}[(n)(n-1)+3(n)+1]a_n x^{n} + \sum_{n=1}^{\infty}[-(n)(n-1)-(n)]a_n x^{n-1} = 0,$$

i.e.,

$$\sum_{n=0}^{\infty}(n+1)^2 a_n x^n - \sum_{n=1}^{\infty} n^2 a_n x^{n-1} = 0, \quad i.e., \quad \sum_{n=0}^{\infty}(n+1)^2 a_n x^n - \sum_{n=0}^{\infty}(n+1)^2 a_{n+1} x^n = 0.$$

Thus, $a_{n+1} = a_n$ for all $n \in \mathbb{N}_0$, and consequently, taking $a_0 = 1$,

$$y_1(x) = \sum_{n=0}^{\infty} x^n = \frac{a_0}{1-x}.$$

Now,

$$y_2(x) = y_1(x)\int \frac{e^{-\int p dx}}{[y_1(x)]^2}dx, \quad p(x) := \frac{3x-1}{x(x-1)}.$$

Note that

$$\begin{aligned}
\int p(x)dx &= \int \frac{3}{x-1}dx - \int \frac{1}{x(x-1)}dx = \int \frac{3}{x-1}dx + \int \frac{1}{x}dx - \int \frac{1}{x-1}dx \\
&= 3\ln|x-1| + \ln|x| - \ln|x-1| = 2\ln|x-1| + \ln|x| = \ln|(x-1)^2 x|,
\end{aligned}$$

$$\frac{e^{-\int p dx}}{[y_1(x)]^2} = \frac{1}{|(x-1)^2 x|[y_1(x)]^2} = \frac{1}{x}.$$

Thus,

$$y_2(x) = \frac{\ln(x)}{1-x}.$$

$\diamondsuit$

Example 4.13. Consider the DE:

$$(x^2-1)x^2y'' - (x^2+1)xy' + (x^2+1)y = 0. \qquad (*)$$

This is of the form (1) with $b(x) = -\dfrac{(x^2+1)}{(x^2-1)}$, $c(x) = \dfrac{x^2+1}{x^2-1}$. Now, taking $y = x^r \sum_{n=0}^{\infty} a_n x^n$, we obtain from (1):

$$\begin{aligned}
(x^2-1)x^2y'' &= (x^2-1)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} \\
&= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r+2} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} \\
(x^2+1)xy' &= (x^2+1)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} \\
&= \sum_{n=0}^{\infty}(n+r)a_n x^{n+r+2} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r}, \\
(x^2+1)y &= \sum_{n=0}^{\infty} a_n x^{n+r+2} + \sum_{n=0}^{\infty} a_n x^{n+r}.
\end{aligned}$$

Thus, $(*)$ takes the form

$$\sum_{n=0}^{\infty}[(n+r)(n+r-1)-(n+r)+1]a_n x^{n+r+2} + \sum_{n=0}^{\infty}[-(n+r)(n+r-1)-(n+r)+1]a_n x^{n+r} = 0. \quad (**)$$

Equating coefficient of x^r to 0, we get the indicial equation as

$$[-r(r-1)-r+1]a_0 = 0, \quad i.e., (r^2-1) = 0.$$

The roots are $r_1 = 1$ and $r_2 = -1$. For $r_1 = 1$, $(**)$ takes the form

$$\sum_{n=0}^{\infty}[(n+1)n-(n+1)+1]a_n x^{n+3} + \sum_{n=0}^{\infty}[-(n+1)n-(n+1)+1]a_n x^{n+1} = 0,$$

i.e.,

$$\sum_{n=0}^{\infty} n^2 a_n x^{n+3} - \sum_{n=0}^{\infty} n(n+2)a_n x^{n+1} = 0, \quad i.e.,$$

This implies $a_1 = 0$ and

$$n^2 a_n - (n+2)(n+4)a_{n+2} = 0 \quad \forall\, n \in \mathbb{N}.$$

Hence, $a_n = 0$ for all $n \in \mathbb{N}$ so that $y(x) = x$. Taking $y_1(x) = x$, we obtain the second solution y_2 as

$$y_2(x) = y_1 \int \frac{e^{-\int p}}{y_1^2},$$

where

$$p = -\frac{x^2+1}{(x^2-1)x} = -\frac{(x^2-1)+2}{x^2-1)x} = -\left[\frac{1}{x} + \frac{2}{(x^2-1)x}\right] = -\left[\frac{1}{x-1} + \frac{1}{x+1} - \frac{1}{x}\right].$$

Hence, $e^{-\int p} = \dfrac{x^2-1}{x}$ so that

$$y_2(x) = y_1 \int \frac{e^{-\int p}}{y_1^2} = x \int \frac{1}{x^2}\left(\frac{x^2-1}{x}\right) dx = x \int \frac{x^2-1}{x^3} dx = x\left(\ln(x) + \frac{1}{2x^2}\right).$$

Thus,

$$y_1 = x, \quad y_2 = x\ln(x) + \frac{1}{2x}$$

are linearly independent solutions. $\diamondsuit$

Remark 4.14. It can be seen that if we take the solution as $y = x^r \sum_{n=0}^{\infty} A_n x^n$ with $r = -1$, then we arrive at $A_n = 0$ so that it violates our requirement, and the resulting expression will not be a solution. $\diamondsuit$

4.3.2 Bessel's equation

Bessel's equation is given by

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

where ν is a non-negative real number. This is a special case of the equation

$$y'' + p(x)y' + q(x)y = 0$$

where p, q are such that $xp(x)$ and $x^2 q(x)$ are analytic at 0, i.e., 0 is a regular singular point. Thus, Frobenius method can be applied.

Taking a solution y of the form $y = x^r \displaystyle\sum_{n=0}^{\infty} a_n x^n$, we have

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=2}^{\infty}(a_{n-2}x^{n+r} - \sum_{n=0}^{\infty}\nu^2 a_n x^{n+r} = 0.$$

Coefficient of x^r is 0 $\iff$ $[r(r-1) + r - \nu^2]a_0 \iff r^2 - \nu^2 = 0.$
Coefficient of x^{r+1} is 0 $\iff$ $[(r+1)^2 - \nu^2]a_1 = 0$
Coefficient of x^{r+n}: $[(n+r)(n+r-1) + (n+r) - \nu^2]a_n + a_{n-2}.$

Thus, roots of the indicial equation are $r_1 = \nu$, $r_2 = -\nu$. Taking $r = r_1 = \nu$, we have $a_1 = 0$ and

$$a_n = -\frac{a_{n-2}}{(n+r)(n+r-1) + (n+r) - \nu^2} = -\frac{a_{n-2}}{n^2 + 2n\nu}, \quad n = 2, 3, \ldots.$$

Hence, $a_{2n-1} = 0$ for all $n \in \mathbb{N}$ and

$$a_{2n} = -\frac{a_{2n-2}}{(2n)^2 + 4n\nu} = -\frac{a_{2n-2}}{2^2 n(n+\nu)}, \quad n \in \mathbb{N}.$$

It is a usual convention to take

$$a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}, \quad \Gamma(\alpha) := \int_0^\infty e^{-t} t^{\alpha-1} dt, \quad \alpha > 0.$$

Recall that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. Then we have

$$a_2 = -\frac{a_0}{2^2(1+\nu)} = -\frac{1}{2^{2+\nu}(\nu+1)\Gamma(\nu+1)} = -\frac{1}{2^{2+\nu}\Gamma(\nu+2)},$$
$$a_4 = -\frac{a_2}{2^2 2(2+\nu)} = (-1)^2\frac{1}{2^{4+\nu}2\Gamma(\nu+3)},$$
$$a_{2n} = \frac{(-1)^n}{2^{2n+\nu}n!\Gamma(\nu+n+1)}.$$

The corresponding solution is

$$J_\nu(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+\nu}n!\Gamma(n+\nu+1)}x^{2n+\nu},$$

which is called the **Bessel function of the first kind** of order ν.

Observe:

- Since the Bessel equation involves only ν^2, it follows that
$$J_{-\nu}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n-\nu}n!\Gamma(n-\nu+1)}x^{2n-\nu}$$
is also a solution.
- If ν is not an integer, then $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent solutions.
- If ν is an integer, then say $\nu = k \in \mathbb{N}$ then
$$J_{-k}(x) = (-1)^k J_k(x) \qquad (*)$$
say $\nu = k \in \mathbb{N}$ then so that J_{-k} and J_k are linearly dependent.

To see the above relation $(*)$, note that

$$\begin{aligned} J_k(x) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+k}n!\Gamma(k+n+1)}x^{2n+k}, \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+k}n!(n+k)!}x^{2n+k}, \end{aligned}$$

Also,

$$J_{-\nu}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n-\nu}n!\Gamma(n-\nu+1)!}x^{2n-\nu}.$$

It can be seen that if $n = 1, 2, \ldots, \nu-1$, then $\Gamma(n-\nu-k) \to \infty$ as $\nu \to n$. Hence for $\nu = -k$, $k \in \mathbb{N}$,

$$\begin{aligned} J_{-k}(x) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n-k}n!\Gamma(n-k+1)}x^{2n-k}, \\ &= \sum_{n=k}^{\infty}\frac{(-1)^n}{2^{2n+k}n!(n-k)!}x^{2n-k} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^{n+k}}{2^{2n+k}(n+k)!n!}x^{2n+k} \\ &= (-1)^k J_k(x). \end{aligned}$$

Now, for an integer k, for obtaining a second solution of the Bessel equation which is linearly independent of J_k, we can use the general method, i.e., write the Bessel equation as

$$y'' + p(x)y' + q(x0y = 0$$

and knowing a solution y_1, obtain $y_2 := y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$. Note that

$$p(x) = \frac{1}{x}, \quad q(x) = \frac{x^2 - k^2}{x^2}.$$

Thus, the second solution according to the above formula is

$$Y_k(x) = J_k(x) \int \frac{dx}{x[J_k(x)]^2}.$$

This is called the **Bessel equation of the second kind of order k.**

Now, we observe few more relations:

1. $(x^\nu J_\nu(x))' = x^\nu J_{\nu-1}(x)$.
2. $(x^{-\nu} J_\nu(x))' = -x^{-\nu} J_{\nu+1}(x)$.
3. $J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$.
4. $J_{\nu-1}(x) - J_{\nu-1}(x) = 2J'_\nu(x)$.

Proofs:
Note that

$$\begin{aligned}
(x^\nu J_\nu(x))' &= \sum_{n=0}^{\infty} (-1)^n \frac{(2n+2\nu)x^{2n+2\nu-1}}{2^{2n+\nu} n! \Gamma(n+\nu+1)} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{2(n+\nu)x^{2n+2\nu-1}}{2^{2n+\nu} n! (n+\nu)\Gamma(n+\nu)} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2\nu-1}}{2^{2n+\nu-1} n! \Gamma(n+\nu)} \\
&= x^\nu \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2\nu-1}}{2^{2n+\nu-1} n! \Gamma(n+\nu)} \\
&= x^\nu J_{\nu-1}(x).
\end{aligned}$$

This proves (1). To prove (2), note that

$$\begin{aligned}
\left(x^{-\nu}J_\nu(x)\right)' &= \sum_{n=1}^{\infty}(-1)^n\frac{2nx^{2n-1}}{2^{2n+\nu}n!\Gamma(n+\nu+1)}\\
&= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{2(n+1)x^{2n+1}}{2^{2n+\nu+2}(n+1)!\Gamma(n+\nu+2)}\\
&= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{2^{2n+\nu+1}n!\Gamma(n+\nu+2)}\\
&= x^{-\nu}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n+\nu+1}}{2^{2n+\nu+1}n!\Gamma(n+\nu+2)}\\
&= -x^{-\nu}J_{\nu+1}(x).
\end{aligned}$$

Proofs of (3) & (4): From (1) and (2),

$$\begin{aligned}
J_{\nu-1}(x)+J_{\nu+1}(x) &= x^{-\nu}\left(x^\nu J_\nu(x)\right)' - x^\nu\left(x^{-\nu}J_\nu(x)\right)'\\
&= x^{-\nu}[x^\nu J_\nu'(x)+\nu x^{\nu-1}J_\nu(x)] - x^\nu[x^{-\nu}J_\nu'(x)-\nu x^{-\nu-1}J_\nu(x)]\\
&= \frac{2\nu}{x}J_\nu(x).
\end{aligned}$$

$$\begin{aligned}
J_{\nu-1}(x)-J_{\nu+1}(x) &= x^{-\nu}\left(x^\nu J_\nu(x)\right)' + x^\nu\left(x^{-\nu}J_\nu(x)\right)'\\
&= x^{-\nu}[x^\nu J_\nu'(x)+\nu x^{\nu-1}J_\nu(x)] + x^\nu[x^{-\nu}J_\nu'(x)-\nu x^{-\nu-1}J_\nu(x)]\\
&= 2J_\nu'(x).
\end{aligned}$$

Using the fact $\Gamma(\frac{1}{2})=\sqrt{\pi}$, it can be shown (verify!) that

$$J_{\frac{1}{2}}=\sqrt{\frac{2}{\pi x}}\sin x, \quad J_{-\frac{1}{2}}=\sqrt{\frac{2}{\pi x}}\cos x.$$

4.4 Orthogonality of functions

Definition 4.15. Functions f and g defined on an interval $[a,b]$ are said to be **orthogonal** with respect to a nonzero weight function w if

$$\int_a^b f(x)g(x)w(x)dx = 0.$$

A sequence (f_n) of functions is said to be an **orthogonal sequence of functions** with respect to w if

$$\int_a^b f_i(x)f_j(x)w(x)dx = 0 \text{ for } i\neq j.$$

[Here, we assume that the above integral exits; that is the case, if for example, they are continuous or bounded and piece-wise continuous.] $\diamondsuit$

Note that

$$\int_0^{2\pi} \sin(nx)\sin(mx)dx = \begin{cases} 0 & \text{if } n \neq m, \\ \pi & \text{if } n \neq m, \end{cases}$$

$$\int_0^{2\pi} \cos(nx)\cos(mx)dx = \begin{cases} 0 & \text{if } n \neq m, \\ \pi & \text{if } n \neq m, \end{cases}$$

$$\int_0^{2\pi} \sin(nx)\cos(mx)dx = 0.$$

Thus, writing

$$f_{2n-2}(x) = \cos(nx), \quad f_{2n-1}(x) = \sin(nx) \quad \text{for} \quad n \in \mathbb{N},$$

then (f_n) is an orthogonal sequence of functions with respect to $w = 1$.

Notation: We shall denote

$$\langle f, g\rangle_w := \int_a^b f_i(x)f_j(x)w(x)dx$$

and call this quantity as the **scalar product** of f and g with respect to w. If $w(x) = 1$ for every $x \in [a, b]$, then we shall denote $\langle f, g\rangle := \langle f, g\rangle_w$. We observe that

- $\langle f, f\rangle_w \geq 0$,
- $\langle f + g, h\rangle_w = \langle f, h\rangle_w + \langle g, h\rangle_w$,
- $\langle cf, f\rangle_w = c\langle f, f\rangle_w$.

If f, g, w are continuous functions, then

- $\langle f, f\rangle_w = 0 \iff f = 0$.

Exercise 4.16. Let $f_1, , \ldots, f_n$ be linearly independent continuous functions. Let $g_1 = f_1$ and for $j = 1, \ldots, n$, define $g_1, \ldots, g_n$ iteratively as follows:

$$g_{j+1} = f_{j+1} - \langle f_{j+1}, g_1\rangle_w g_1 - \langle f_{j+1}, g_2\rangle_w g_2 - \cdots \langle f_{j+1}, g_j\rangle_w g_j, \quad j = 1, \ldots, n-1$$

i.e., $g_{j+1} = f_{j+1} - \sum_{i=1}^{j}\langle f_{j+1}, f_i\rangle_w f_i, \quad j = 1, 2, \ldots, n-1$. Prove that $g_1, \ldots, g_n$ are orthogonal functions with respect to w. $\diamondsuit$

Definition 4.17. Functions $f_1, f_2, \ldots$ are said to be **linearly independent** if for every $n \in \mathbb{N}$, $f_1, \ldots, f_n$ are linearly independent, i.e., for every $n \in \mathbb{N}$, if $\alpha_1, \ldots, \alpha_n$ are scalars such that $\alpha_1 f_1 + \cdots + \alpha_n f_n = 0$, then $\alpha_i = 0$ for $i = 1, \ldots, n$. $\diamondsuit$

Definition 4.18. A sequence (f_n) on $[a, b]$ is said to be an **orthonormal sequence** of functions with respect to w if (f_n) is an orthogonal sequence with respect to w and $\langle f_n, f_n\rangle_w = 1$ for every $j \in \mathbb{N}$. $\diamondsuit$

Exercise 4.19. Let $f_j(x) = x^{j-1}$ for $j \in \mathbb{N}$. Find $g_1, g_2, \ldots$ as per the formula in Exercise 4.16 with $w(x) = 1$ and $[a, b] = [-1, 1]$. Observe that, for each $n \in \mathbb{N}$, g_n is a scalar multiple of the Legendre polynomial P_{n-1}. $\diamondsuit$

4.4.1 Orthogonality of Legendre polynomials

Recall that for non-negative integers n, the Legendre equation is given by

$$(1-x^2)y'' - 2xy' + \lambda_n y = 0, \quad \lambda_n := n(n+1).$$

This equation can be written as:

$$[(1-x^2)y']' + \lambda_n y = 0. \tag{$*$}$$

Recall that for each $n \in \mathbb{N}_0$, the Legendre polynomial

$$P_n(x) = \sum_{k=0}^{M_n} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}, \quad M_n := \begin{cases} \frac{n}{2} & \text{if } n \text{ even,} \\ \frac{n-1}{2} & \text{if } n \text{ odd} \end{cases}$$

satisfies the equation $(*)$. Thus,

$$[(1-x^2)P_n']' + \lambda_n P_n = 0, \tag{$*_1$}$$

$$[(1-x^2)P_m']' + \lambda_m P_m = 0. \tag{$*_2$}$$

$\Longrightarrow$

$$[(1-x^2)P_n']'P_m + \lambda_n P_n P_m = 0, \quad [(1-x^2)P_m']'P_n + \lambda_m P_m P_n = 0$$

$\Longrightarrow$

$$\{[(1-x^2)P_n']'P_m - [(1-x^2)P_m']'P_n\} + (\lambda_n - \lambda_m)P_n P_m = 0,$$

i.e.,

$$[(1-x^2)P_n'P_m]' - [(1-x^2)P_m'P_n]' + (\lambda_n - \lambda_m)P_n P_m = 0$$

$\Longrightarrow$

$$\int_{-1}^{1} \{[(1-x^2)P_n'P_m]' - [(1-x^2)P_m'P_n]'\}dx + (\lambda_n - \lambda_m) \int_{-1}^{1} P_n P_m dx = 0$$

i.e.,

$$(\lambda_n - \lambda_m) \int_{-1}^{1} P_n P_m dx = 0.$$

Thus,

$$n \neq m \implies \lambda_n \neq \lambda_m \implies \int_{-1}^{1} P_n P_m dx = 0.$$

Using the expression for P_n, it can be shown that

$$\int_{-1}^{1} P_n^2 dx = \frac{2}{2n+1}.$$

Hence,

- $\left\{ \sqrt{\frac{2n+1}{2}}\, P_n : n \in \mathbb{N}_0 \right\}$ is an orthonormal sequence of polynomials.

Remark 4.20. Recall that for $n \in \mathbb{N}_0$, the Legenendre polynomial $P_n(x)$ is of degree n and the $P_0, P_1, P_2, \ldots$ are orthogonal. Hence $P_0, P_1, P_2, \ldots$ are linearly independent. We recall the following result from *Linear Algebra*:

- If $q_0, q_1, \ldots, q_n$ are polynomials which are
    1. linearly independent and
    2. degree of q_j is atmost n for each $j = 0, 1, \ldots, n$,

  then every polynomial q of degree at most n can be uniquely represented as
  $$q = c_0 q_0 + c_1 q_1 + \ldots + c_n q_n.$$

In the above if $q_0, q_1, \ldots, q_n$ are orthogonal also, i.e., $\langle q_j, q_k \rangle = 0$ for $j \neq k$, then we obtain
$$c_j = \frac{\langle q, q_j \rangle}{\langle q_j, q_j \rangle}, \quad j = 0, 1, \ldots, n.$$
Thus,
$$q = \sum_{j=0}^{n} c_j q_j = \sum_{j=0}^{n} \frac{\langle q, q_j \rangle}{\langle q_j, q_j \rangle} q_j.$$
In particular:

- If q is a polynomial of degree n, then
  $$q = \sum_{j=0}^{n} \frac{\langle q, P_j \rangle}{\langle P_j, P_j \rangle} P_j,$$
  where $P_0, P_1, \ldots$ are Legendre polynomials.

From *Real Analysis*, we recall that:

- For every continuous function f defined on a closed and bounded interval $[a, b]$, there exists a sequence (q_n) of polynomials such that (q_n) converges to f uniformly on $[a, b]$, i.e., for every $\varepsilon > 0$ there exists a positive integer N_ε such that
  $$|f(x) - q_n(x)| \leq \varepsilon \quad \forall\, n \geq N_\varepsilon, \quad \forall\, x \in [a, b].$$

The above result is known as *Weierstrass approximation theorem.* Using the above result it can be shown that:

- If $q_0, q_1, \ldots,$ are nonzero orthogonal polynomials on $[a, b]$ such that $\max\limits_{0 \leq j \leq n} \deg(q_j) \leq n$, then every continuous function f defined on $[a, b]$ can be represented as
  $$f = \sum_{j=0}^{\infty} c_j q_j, \quad c_j := \frac{\langle q, q_j \rangle}{\langle q_j, q_j \rangle}, \quad j \in \mathbb{N}_0. \tag{$*$}$$
  The equality in the above should be understood in the sense that
  $$\|f - \sum_{j=n}^{\infty} c_j q_j\| \to 0 \quad \text{as} \quad n \to \infty$$
  where $\|g\|^2 := \langle g, g \rangle$.

The expansion in $(*)$ above is called the **Fourier expansion of** f with respect to the orthogonal polynomials q_n, $n \in \mathbb{N}_0$. If we take $P_0, P_1, P_2, \ldots$ on $[-1,1]$, then the corresponding Fourier expansion is known as **Fourier–Legendre expansion**. $\diamondsuit$

4.4.2 Orthogonal polynomials defined by Bessel functions

Recall that for a positive integer $n \in \mathbb{N}$, the Bessel function of the first kind of order n is given by

$$J_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{2j+n} j! \Gamma(n+j+1)} x^{2j+n}$$

is a power series, and it satisfies he Bessel equation:

$$x^2 J_n'' + x J_n' + (x^2 - n^2) J_n = 0.$$

THEOREM 4.21. *If α and β are zeros of $J_n(x)$ in the interval $[0,1]$, then*

$$\int_0^1 x J_n(\alpha x) J_n(\beta x) dx = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ \frac{1}{2} J_{n+1}(\alpha), & \text{if } \alpha = \beta. \end{cases}$$

Proof. Observe that, for $\lambda \in \mathbb{R}$, if $z = \lambda x$ and $y(x) = J_n(\lambda x)$, then

$$y_n'(x) = \lambda J_n'(\lambda x) = \lambda J_n(z), \quad y_n''(x) = \lambda^2 J_n''(z).$$

Thus, we have

$$z^2 J_n''(z) + z J_n'(z) + (z^2 - n^2) J_n(z) = 0 \iff \lambda^2 x^2 \frac{y_n''(x)}{\lambda^2} + \lambda x \frac{y_n'(x)}{\lambda} + (\lambda^2 x^2 - n^2) y_n(x) = 0$$

$\iff$

$$x^2 y_n''(x) + x y_n'(x) + (\lambda^2 x^2 - n^2) y_n(x) = 0$$

Now, let

$$u(x) = J_n(\alpha x), \quad v(x) = J_n(\beta x).$$

Thus, we have

$$x^2 u'' + x u' + (\alpha^2 x^2 - n^2) u = 0, \qquad x^2 v'' + x v' + (\beta^2 x^2 - n^2) v = 0$$

$\iff$

$$x u'' + u' + (\alpha^2 x - \frac{n^2}{x}) u = 0, \qquad x v'' + v' + (\beta^2 x - \frac{n^2}{x}) v = 0$$

$\implies$

$$v\Big[x u'' + u' + (\alpha^2 x - \frac{n^2}{x}) u\Big] = 0, \qquad u\Big[x v'' + v' + (\beta^2 x - \frac{n^2}{x}) v\Big] = 0$$

$\implies$

$$x[v u'' - u v''] + [v u' - u v'] + (\alpha^2 - \beta^2) x u v = 0$$

$\Longleftrightarrow$

$$\frac{d}{dx}[x(vu' - uv')] + (\alpha^2 - \beta^2)xuv = 0$$

$\Longrightarrow$

$$\int_0^1 \frac{d}{dx}[x(vu' - uv')]dx + (\alpha^2 - \beta^2)\int_0^1 xuvdx = 0.$$

Since $u(1) = J_n(\alpha) = 0$ and $v(1) = J_n(\beta) = 0$, it follows that

$$(\alpha^2 - \beta^2)\int_0^1 xuvdx = 0.$$

Hence,

$$\alpha \neq \beta \quad \Longrightarrow \quad \int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = 0.$$

Next, we consider the case of $\beta = \alpha$: Note that

$$2u'[x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0,$$

i.e.,

$$2x^2u'u'' + 2xu'u' + 2(\alpha^2x^2 - n^2)u'u = 0,$$

i.e.,

$$[x^2(u')^2]' + 2(\alpha^2x^2 - n^2)u'u = 0,$$

Also,

$$[\alpha^2x^2u^2 - n^2u^2]' = \alpha^2(2x^2uu' + 2xu^2) - n^2(2uu') = 2(\alpha^2x^2 - n^2)u'u + 2\alpha^2xu^2.$$

Thus,

$$[x^2(u')^2]' + 2(\alpha^2x^2 - n^2)u'u = 0$$

$\Longleftrightarrow$

$$[x^2(u')^2]' + [\alpha^2x^2u^2 - n^2u^2]' - 2\alpha^2xu^2 = 0,$$

$\Longrightarrow$

$$\int_0^1 [x^2(u')^2]'dx + \int_0^1 [\alpha^2x^2u^2 - n^2u^2]'dx - 2\alpha^2\int_0^1 xu^2dx = 0,$$

i.e.,

$$[x^2(u')^2]_0^1 + [\alpha^2x^2u^2 - n^2u^2]_0^1 - 2\alpha^2\int_0^1 xu^2dx = 0,$$

Since $u(1) = J_n(\alpha) = 0$ and $u(0) = J_n(0) = 0$, it follows that

$$[u'(1)]^2 - 2\alpha^2\int_0^1 xu^2dx = 0,$$

i.e.,

$$\int_0^1 x[J_n(\alpha x)]^2dx = \frac{1}{2}[J_n'(\alpha)]^2 = \frac{1}{2}J_{n+1}(\alpha).$$

The last equality follows, since:

$$(x^{-n}J_n)' = -x^{-n}J_{n+1} \Longleftrightarrow x^{-n}J_n' - nx^{-n-1}J_n = -x^{-n}J_{n+1}$$

so that taking $x = \alpha$,

$$-\alpha^{-n} J_{n+1}(\alpha) = \alpha^{-n} J_n'(\alpha) - n\alpha^{-n-1} J_n(\alpha) = \alpha^{-n} J_n'(\alpha).$$

Thus, $J_n'(\alpha) = J_{n+1}(\alpha)$, and the proof is complete. $\square$

5 Sturm–Liouville problem (SLP)

Definition 5.1. For continuous real valued functions p, q, r defined on interval such that r' exists and continuous and $p(x) > 0$ for all $x \in [a, b]$, consider the differential equation

$$(r(x)y')' + [q(x) + \lambda p(x)]y = 0, \tag{1}$$

together with the *boundary conditions*

$$k_1 y(a) + k_2 y'(a) = 0, \tag{2}$$

$$\ell_1 y(b) + \ell_2 y'(b) = 0. \tag{3}$$

The problem of determining a scalar λ and a corresponding nonzero function y satisfying (1)–(3) is called a **Sturm–Liouville problem (SLP)**. A scalar (real or complex number) λ for which there is a nonzero function y satisfying (1)–(3) is called an **eigenvalue** of the SLP, and in that case the function y is called the corresponding **eigenfunction**. $\diamondsuit$

We assume the following known result.

THEOREM 5.2. *Under the assumptions on p, q, r given in Definition 5.1, the set of all eigenvalues of SLP is a countably infinite set*[4].

THEOREM 5.3. *Eigenfunctions corresponding to distinct eigenvalues are orthogonal on* $[a, b]$ with respect to *the weight function $p(x)$.*

Proof. Suppose λ_1 and λ_2 are eigenvalues of the SLP with corresponding eigenvectors y_1 and y_2, respectively. Let us denote

$$Ly := [r(x)y']' + q(x)y.$$

Then we have Let us denote

$$Ly_1 = -\lambda_1 p y_1, \qquad Ly_2 = -\lambda_2 p y_2.$$

$\Longrightarrow$

$$(Ly_1)y_2 - (Ly_2)y_1 = (\lambda_2 - \lambda_1) p y_1 y_2.$$

$\Longrightarrow$

$$\int_a^b [(Ly_1)y_2 - (Ly_2)y_1 dx = (\lambda_2 - \lambda_1) \int_a^b p y_1 y_2 dx.$$

Note that

$$(Ly_1)y_2 - (Ly_2)y_1 = [(ry_1')y_2 - (ry_2')y_1]'.$$

[4] A set S is said to be *countably infinite* if it is in one-one corresponding to the set $\mathbb{N}$ of natural numbers. For example, other than $\mathbb{N}$ itself, the set $\mathbb{Z}$ of all integers, and the set $\mathbb{Q}$ of all rational numbers are countably infinite. However, the set $\{x \in \mathbb{R} : 0 < x < 1\}$ is not a countably infinite set. An infinite set which is not countably infinite is called an *uncountable set*. For example, the set $\{x \in \mathbb{R} : 0 < x < 1\}$ is an uncountable set; so also the set of all irrational numbers in $\{x \in \mathbb{R} : 0 < x < 1\}$

Hence

$$\int_a^b [(Ly_1)y_2 - (Ly_2)y_1 dx = [(ry_1')y_2 - (ry_2')y_1](b) - [(ry_1')y_2 - (ry_2')y_1](a).$$

Using the boundary conditions, the last expression on the above can be shown to be 0. Thus, we obtain

$$(\lambda_2 - \lambda_1) \int_a^b py_1y_2\, dx = [(ry_1')y_2 - (ry_2')y_1](b) - [(ry_1')y_2 - (ry_2')y_1](a) = 0.$$

Therefore, if $\lambda_2 \neq \lambda_1$, we obtain $\int_a^b py_1y_2\, dx = 0.$ □

THEOREM 5.4. *Every eigenvalue of the SLP (1)–(3) is real.*

Proof. Let us denote

$$Ly := [r(x)y']' + q(x)y.$$

Suppose $\lambda := \alpha + i\beta$ is an eigenvalue of SLP with corresponding eigenfunction $y(x) = u(x) + iv(x)$, where $\alpha, \beta \in \mathbb{R}$, and u, v are real valued functions. Then we have

$$L(u + iv) = -(\alpha + i\beta)p(u + iv),$$

i.e.,

$$Lu + iLv = -p(\alpha u - \beta v) - ip(\alpha v + \beta u).$$

Hence,

$$Lu = -p(\alpha u - \beta v), \qquad Lv = -p(\alpha v + \beta u)$$

$\Longrightarrow$

$$(Lu)v - (Lv)u = \beta p(v^2 + u^2).$$

$\Longrightarrow$

$$\int_a^b [(Lu)v - (Lv)u]dx = \beta \int_a^b p(v^2 + u^2)dx.$$

But,

$$(Lu)v - (Lv)u = [(ru')v - (rv')u]'.$$

Hence,

$$\int_a^b [(Lu)v - (Lv)u]dx = \int_a^b [(ru') - (rv')u]'dx = [(ru')v - (rv')u](b) - [(ru')v - (rv')u](a).$$

Using the fact that u and v satisfy the boundary conditions (2)-(3), it can be shown that

$$[(ru')v - (rv')u](b) - [(ru')v - (rv')u](a) = 0.$$

Thus, we obtain $\beta \int_a^b p(v^2 + u^2)dx = 0$. Since $\beta \int_a^b p(v^2 + u^2)dx$ we obtain $\beta = 0$, and hence $\lambda = \alpha \in \mathbb{R}$. □

THEOREM 5.5. *If y_1 and y_2 are the eigenfunctions corresponding to an eigenvalue λ of the SLP, then prove that y_1, y_2 are linearly dependent.*

Proof. Suppose y_1 and y_2 are eigenfunctions corresponding to an eigenvalue λ of the SLP. Then we have

$$Ly_1 = -\lambda p y_1, \quad Ly_2 = -\lambda p y_2.$$

Hence,

$$(Ly_1)y_2 - (Ly_2)y_1 = 0.$$

But,

$$(Ly_1)y_2 - (Ly_2)y_1 = [(ry_1')y_2 - (ry_2')y_1]' = [rW(y_1, y_2)]'.$$

Thus $[rW(y_1, y_2)]' = 0$ so that, using the assumption that r is not a zero function, we obtain $rW(y_1, y_2)$ is a constant function, say

$$r(x)W(y_1, y_2)(x) = c, \text{ constant}.$$

But, by the boundary condition (2) we have

$$\begin{aligned} k_1 y_1(a) + k_2 y_1'(a) &= 0 \\ k_1 y_2(a) + k_2 y_2'(a) &= 0 \end{aligned}$$

i.e.,

$$\begin{bmatrix} y_1(a) & y_1'(a) \\ y_2(a) & y_2'(a) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence, $W(y_1, y_2)(a) = 0$ so that $r(a)W(y_1, y_2)(a) = 0$ and hence, $c = 0$. This implies that $W(y_1, y_2)$ is a zero function, and hence y_1, y_2 are linearly dependent. □

Example 5.6. For $\lambda \in \mathbb{R}$, consider the SLP:

$$y'' + \lambda y = 0, \qquad y(0) = 0 = y(\pi)$$

Note that, for $\lambda = 0$, the problem has only zero solution. Hence, 0 is not an eigenvalue of the problem.

If $\lambda < 0$, say $\lambda = -\mu^2$, then a general solution is given by

$$y(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}.$$

Now, $y(0)$ implies $C_1 + C_2 = 0$ and $y(\pi) = 0$ implies $C_1 e^{i\mu\pi} + C_1 e^{-i\mu\pi} = 0$. Then, it follows that, $C_1 = 0 = C_2$. Hence, the SLP does not have any negative eigenvalues.

Next suppose that $\lambda > 0$, say $\lambda = \mu^2$. Then a general solution is given by

$$y(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x).$$

Note that $y(0) = 0$ implies $C_1 = 0$. Now, $y(\pi) = 0$ implies $y(\pi) = C_2 \sin(\mu\pi) = 0$. Hence, for those values of μ for which $\sin(\mu\pi) = 0$, we obtain nonzero solution. Now,

$$\sin(\mu\pi) = 0 \iff \mu\pi = n\pi \quad \text{for} \quad n \in \mathbb{Z}.$$

Thus the eigenvalues and corresponding eigenfunctions of the SLP are

$$\lambda_n := n^2, \quad y_n(x) := \sin(nx),\ n \in \mathbb{N}.$$

◇

Example 5.7. For $\lambda \in \mathbb{R}$, consider the SLP:

$$y'' + \lambda y = 0, \qquad y'(0) = 0 = y'(\pi)$$

Note that, for $\lambda = 0$, $y(x) = \alpha + \beta x$ is a solution of the DE. Now, $y'(0) = 0 = y'(\pi) = 0$ imply $\beta = 0$. Hence, $y(x) = 1$ is a solution.

If $\lambda < 0$, say $\lambda = -\mu^2$, then a general solution is given by

$$y(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}.$$

Note that $y'(x) = \mu C_1 e^{\mu x} - \mu C_2 e^{-\mu x}$. Hence,

$$y'(0) = 0 = y'(\pi) \implies C_1 - C_2 = 0, \quad C_1 e^{\mu\pi} - C_2 e^{-\mu\pi} = 0.$$

Hence, $C_1 = C_2 = 0$, and hence the SLP does not have any negative eigenvalues.

Next suppose that $\lambda > 0$, say $\lambda = \mu^2$. Then a general solution is given by

$$y(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x).$$

Then,

$$y'(x) = -\mu C_1 \sin(\mu x) + \mu C_2 \cos(\mu x).$$

Now, $y(0)$ implies $C_2 = 0$, and hence $y(\pi) = 0$ implies $sin(\mu\pi) = 0$. Note that

$$\sin(\mu\pi) = 0 \iff \mu\pi = n\pi \quad \text{for} \quad n \in \mathbb{Z}.$$

Thus the eigenvalues and corresponding eigenfunctions of the SLP are

$$\lambda_n := n^2, \quad y_n(x) := \cos(nx),\ n \in \mathbb{N}_0.$$

◇

Exercise 5.8. For $\lambda \in \mathbb{R}$, consider the SLP:

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(\pi) = 0.$$

Show that the eigenvalues and the corresponding eigenfunctions for the above SLP are given by

$$\lambda_n = \left(\frac{2n-1}{2}\right)^2, \quad y_n(x) = \sin\left[\left(\frac{2n-1}{2}\right)x\right], \quad n \in \mathbb{N}.$$

◇

Exercise 5.9. Consider the Schrödinger equation:

$$-\frac{h^2}{2\pi m}\psi''(x) = \lambda\psi x, \quad x \in [0, \ell],$$

along with the boundary condition

$$\psi(0) = 0 = \psi(\ell).$$

Show that the eigenvalues and the corresponding eigenfunctions for the above SLP are given by

$$\lambda_n = \frac{h^2\pi^2 n^2}{2m\ell^2}, \qquad \psi_n(x) = \sqrt{\frac{2}{\ell}}\sin\left(\frac{n\pi x}{\ell}\right), \quad n \in \mathbb{N}.$$

◇

Exercise 5.10. Let

$$Ly := [r(x)y']' + q(x)y.$$

Prove that

$$\langle Ly, z\rangle_p = \langle y, Lz\rangle_p \quad \forall\, y, z \in C[a, b],$$

for every weight function $p(x) > 0$ on $[a, b]$. ♢

Definition 5.11. An orthogonal sequence (φ_n) of nonzero functions in $C[a, b]$ is called a *complete system* for $C[a, b]$ with respect to a weight function w if every $f \in C[a, b]$ can be written as

$$f = \sum_{n=1}^{\infty} c_n \varphi_n,$$

where the equality above is in the sense that

$$\int_a^b \left| f(x) - \sum_{n=1}^{N} c_n \varphi_n(x) \right|^2 w(x)dx \to 0 \quad \text{as} \quad N \to \infty.$$

It can be seen that $c_n = \dfrac{\langle f, \varphi_n\rangle_w}{\langle f_n, \varphi_n\rangle_w}$. ♢

References

[1] William E. Boycee and Richard C. DiPrima (2012): *Elementary Differential Equations* , John Wiley and Sons, Inc.