Chapter 2. Boundary-Value Problems in Electrostatics: Method of Images and Expansion in Orthogonal Functions

2.1 Method of images

The method of images is based on the uniqueness theorem: for a given set of boundary conditions the solution to the Poisson's equation is unique. Figure 2.1 compares two different external charge distributions, (a) ρ, σ and (b) q, q', q'', … , but the boundary conditions of the two systems are identical. Then, the potentials are same inside the regions. The image charges must be external to the volume of interest.
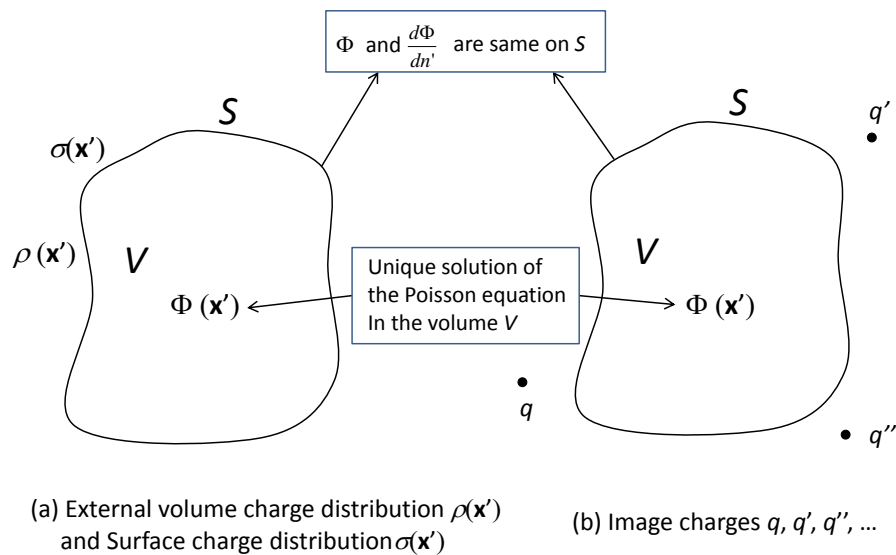
Φ and $\frac{d\Phi}{dn'}$ are same on S

S

$\sigma(\mathbf{x}')$

q'

$\rho(\mathbf{x}')$ V

V

Unique solution of the Poisson equation In the volume V

$\Phi(\mathbf{x}') \leftarrow$

$\rightarrow \Phi(\mathbf{x}')$

q

q''

(a) External volume charge distribution $\rho(\mathbf{x}')$ and Surface charge distribution $\sigma(\mathbf{x}')$

(b) Image charges q, q', q'', …

Fig 2.1. Method of images

Point charge and conducting plane

(a)

(b)

$\Phi=0$

$\Phi=0$

$\mathbf{x}_2$

$\mathbf{x}_1$

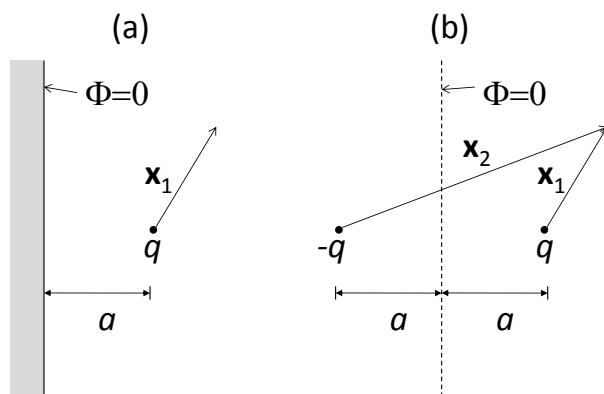$\mathbf{x}_1$

q

$-q$

q

a

a

a

Fig 2.2. Problem of a point charge and conducting plane solved by means of the image-charge method: (a) original problem, (b) location of image charge, where $|\mathbf{x}_1| = r_1$, $|\mathbf{x}_2| = r_2$

A point charge q is placed near a conducting plane of infinite extent (see Fig.2.2a). The boundary condition is that $\Phi = 0$ on the surface of the conducting plane. Let the conducting plane coincide with the *yz*-plane and the point charge line on the *x*-axis at *x=a*. Consider now a system of two point charges a distance *2a* apart as shown in Fig. 2.2b.

Potential

The potential of the two charges ,

$$\Phi(x,y,z) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_1} - \frac{1}{4\pi\varepsilon_0}\frac{q}{r_2}$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+y^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+y^2+z^2}}\right], \qquad (2.1)$$

satisfies not only (i) Poisson equation for x>0 and (ii) the boundary at all points exterior to the charges, but also the boundary condition of the original problem. Therefore, Eq. 2.1 is the correct potential in the entire half-space exterior to the conducting plane (*x>0*).

Induced Surface charge

The surface charge density induced on the conductor

$$\sigma(y,z) = \epsilon_0 E_x|_{x=0} = -\epsilon_0 \frac{\partial\Phi}{\partial x}\bigg|_{x=0} = -\frac{qa}{2\pi(a^2 + y^2 + z^2)^{3/2}} \qquad (2.2)$$

The total charge induced on the plane is

$$\iint \sigma(y,z)\,dydz = \int_0^{2\pi}\int_0^\infty \frac{-qa}{2\pi(a^2 + r^2)^{3/2}}r\,dr\,d\phi = \frac{qa}{\sqrt{r^2 + a^2}}\bigg|_0^\infty = -q \qquad (2.3)$$

where $y^2 + z^2 = r^2$. The charge *q* is attracted toward the plane because of the negative induced charge. The force acting on the charge is

$$\mathbf{F} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2a)^2}\mathbf{e}_x \qquad (2.4)$$
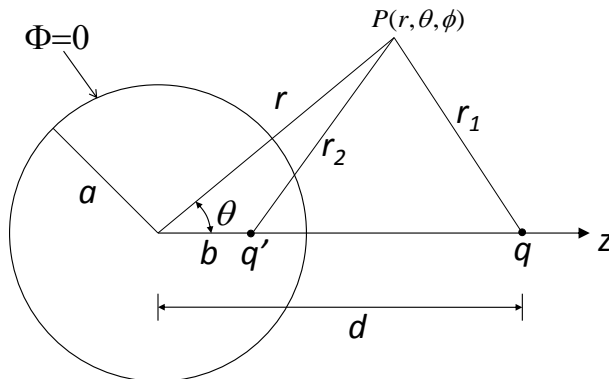
Point charge and conducting sphere

Fig 2.3. Point charge *q* (at *z=d*) in the vicinity of a grounded conducting sphere (radius *a*); *q'* is the image charge (at *z=b*).

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Figure 2.3 illustrates a point charge in the vicinity of a grounded conducting sphere of radius a. It is convenient to formulate the problem by means of spherical coordinates, with the origin of coordinates at the center of the sphere. Let the charge q at $z=d$ on the z-axis. The boundary condition, $\Phi(r = a) = 0$, can be satisfied by an image charge q' inside the sphere ($z=b$).

Potential
The potential due to the charges q and q'

$$\Phi(r, \theta, \phi) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_1} + \frac{1}{4\pi\varepsilon_0}\frac{q'}{r_2}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{r^2+d^2-2rd\cos\theta}} + \frac{q'}{\sqrt{r^2+b^2-2rb\cos\theta}}\right], \qquad (2.5)$$

$\Phi(r = a, \theta, \phi)$ can equal zero for all θ only if $\sqrt{\dfrac{a^2+b^2-2ab\cos\theta}{a^2+d^2-2ad\cos\theta}} = -\dfrac{q'}{q} = $ constant. This is

the case if $b = \dfrac{a^2}{d}$, for then $\sqrt{\dfrac{a^2+b^2-2ab\cos\theta}{a^2+d^2-2ad\cos\theta}} = \dfrac{a}{d}$. Hence,

$$b = \frac{a^2}{d} \qquad (2.6) \qquad \text{and} \qquad q' = -\frac{a}{d}q \qquad (2.7)$$

These equations specify the location and magnitude of the image charge q'.

Induced Surface charge
The surface charge density induced on the conductor

$$\sigma(\theta, \phi) = -\epsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = -\frac{q}{4\pi a^2}\left(\frac{a}{d}\right)\frac{1 - \dfrac{a^2}{d^2}}{\left(1 + \dfrac{a^2}{d^2} - 2\dfrac{a}{d}\cos\theta\right)^{3/2}} \qquad (2.8)$$

The total charge induced on the plane is

$$\int_0^{2\pi}\int_0^{\pi}\sigma(\theta,\phi)\,a^2\sin\theta\,d\theta d\phi = -\frac{a}{d}q = q' \qquad (2.9)$$

The charge q is attracted toward the sphere because of the negative induced charge. The force acting on the charge is

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(d - b)^2}\mathbf{e}_z = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{a^2}\left(\frac{a}{d}\right)^3\left(1 - \frac{a^2}{d^2}\right)^{-2}\mathbf{e}_z \qquad (2.10)$$

Conducting sphere of nonzero potential
A second image charge q'' may be placed at the center of the sphere without destroying the equipotential nature of the spherical surface. The magnitude of q'' is arbitrary; it may be

adjusted to fit the boundary conditions on the problem. The potential at all points exterior to the sphere is

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1} + \frac{q'}{r_2} + \frac{q''}{r}\right) \tag{2.11}$$

The potential of the spherical conductor itself is

$$\Phi(a,\theta,\phi) = \frac{1}{4\pi\varepsilon_0}\frac{q''}{a} \tag{2.12}$$

The total charge on the sphere Q is equal to the sum of q' and q''.

- For an uncharged spherical conductor, Q=0, therefore, $q'' = -q' = \frac{a}{d}q$.
- For a biased spherical conductor, $\Phi(a,\theta,\phi) = V$, $q'' = 4\pi\varepsilon_0 aV$.
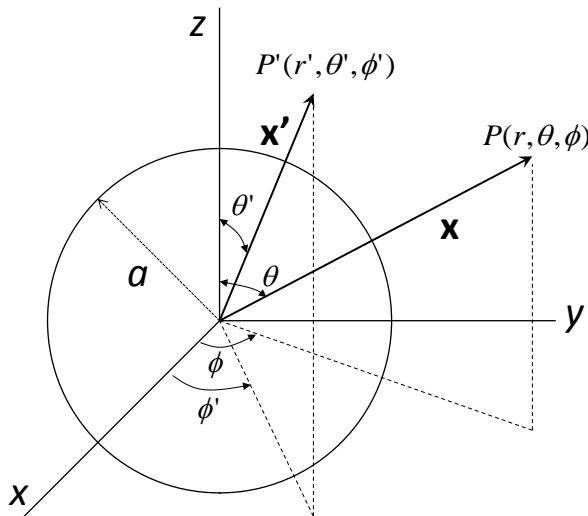
Green function for the Sphere

Fig 2.4. x' refers to the location *P'* of the unit source and the variable x is the point *P* at which the potential is being evaluated, or vise versa.

The Green function appropriate for Dirichlet boundary conditions on the sphere of radius a is expressed as

$$G(\mathbf{x},\mathbf{x}') = G(\mathbf{x}',\mathbf{x}) = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + F(\mathbf{x},\mathbf{x}') \tag{2.13}$$

where the following conditions must be satisfied: (1) $\nabla'^2 F(\mathbf{x},\mathbf{x}') = 0$ for $r' > a$ and (2) $G(\mathbf{x},\mathbf{x}') = 0$ for $r' = a$. The discussion of the conducting sphere with the method of images indicates that

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{a}{r'\left|\mathbf{x}-\frac{a^2}{r'^2}\mathbf{x}'\right|} \tag{2.14}$$

meets all the Green function requirements. In terms of spherical coordinates it can be written:

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$$G(\mathbf{x}, \mathbf{x}') = G(\mathbf{x}', \mathbf{x}) = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \gamma}} - \frac{1}{\sqrt{\frac{r^2 r'^2}{a^2} + a^2 - 2rr' \cos \gamma}} \tag{2.15}$$

where γ is the angle between $\mathbf{x}$ and $\mathbf{x}'$. $G = 0$ if either $\mathbf{x}$ or $\mathbf{x}'$ is on the surface of the sphere. The solution of the Laplace equation (see Eq. 1.30 with $\rho(\mathbf{x}') = 0$ outside the sphere)

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi} \int_S \left[\Phi(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} \right] da', \text{ where } \frac{\partial}{\partial n'} = -\frac{\partial}{\partial r'}$$

$$= \frac{1}{4\pi} \int \Phi(a, \theta', \phi') \frac{a(r^2 - a^2)}{(r^2 + a^2 - 2ar \cos \gamma)^{3/2}} d\Omega' \tag{2.16}$$

Where $d\Omega' = \sin \theta' \, d\theta' d\phi'$ is the element of solid angle at the point (a, θ', ϕ') and $\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$.

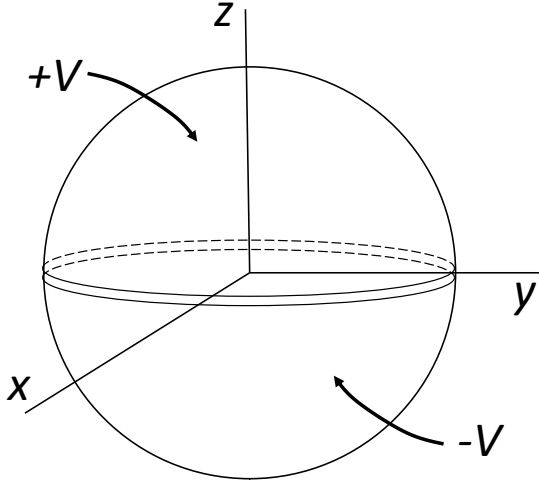Conducting sphere with hemispheres at different potentials

Fig 2.5. Conducting sphere of radius a made up of two hemispherical shells separated by a thin insulating ring. They are kept at different potentials, $+V$ and $-V$.

As an example of the solution Eq. 2.16, we consider a conducting sphere made up of two hemispherical shells kept at their potentials of $+V$ and $-V$ as shown in Fig. 2.5.

$$\Phi(r, \theta, \phi) = \frac{V}{4\pi} \int_0^{2\pi} d\phi' \left[\int_0^{\frac{\pi}{2}} \sin \theta' \, d\theta' - \int_{\frac{\pi}{2}}^{\pi} \sin \theta' \, d\theta' \right] \frac{a(r^2 - a^2)}{(r^2 + a^2 - 2ar \cos \gamma)^{3/2}}$$

$$= \frac{V}{4\pi} \int_0^{2\pi} d\phi' \left[\int_0^1 d \cos \theta' - \int_{-1}^0 d \cos \theta' \right] \frac{a(r^2 - a^2)}{(r^2 + a^2 - 2ar \cos \gamma)^{\frac{3}{2}}} \tag{2.17}$$

By changing the variable from $\mathbf{x}'$ to $-\mathbf{x}'$ in the second integral, we obtain

$$\Phi(r, \theta, \phi) = \frac{Va(r^2 - a^2)}{4\pi} \int_0^{2\pi} d\phi' \int_0^1 d \cos \theta' \left[\frac{1}{(r^2 + a^2 - 2ar \cos \gamma)^{\frac{3}{2}}} - \frac{1}{(r^2 + a^2 + 2ar \cos \gamma)^{\frac{3}{2}}} \right]$$

$$\tag{2.18}$$

- Potential on the positive z axis ($r = z > 0$): $\theta = 0$, hence $\cos \gamma = \cos \theta' \equiv t$

$$\Phi(z) = \frac{Va(z^2-a^2)}{2} \int_0^1 \left[\frac{1}{(z^2+a^2-2azt)^{\frac{3}{2}}} - \frac{1}{(z^2+a^2+2azt)^{\frac{3}{2}}} \right] dt$$

$$= V \left[1 - \frac{z^2-a^2}{z\sqrt{z^2+a^2}} \right] \tag{2.19}$$

At $z = a$, $\Phi = V$, and as $z \to \infty$, $\Phi \cong 3Va^2/2z^2$.

- In the far field, i.e., for $r \gg a$, $\alpha = \frac{ar}{a^2+r^2} \cong \frac{a}{r} \ll 1$. Then, Eq. 2.18 becomes

$$\Phi(r, \theta, \phi) = \frac{Va(r^2-a^2)}{4\pi(r^2+a^2)^{\frac{3}{2}}} \int_0^{2\pi} d\phi' \int_0^1 dt \left[\frac{1}{(1-2\alpha \cos \gamma)^{\frac{3}{2}}} - \frac{1}{(1+2\alpha \cos \gamma)^{\frac{3}{2}}} \right]$$

$$\cong \frac{Va}{4\pi r} \int_0^{2\pi} \int_0^1 6\alpha \cos \gamma \, dt \, d\phi'$$

$$= \frac{3Va^2}{2\pi r^2} \int_0^{2\pi} \int_0^1 \left[t \cos \theta + \sqrt{1 - t^2} \sin \theta \cos(\phi - \phi') \right] dt \, d\phi'$$

$$= \frac{3Va^2}{r^2} \cos \theta \int_0^1 t \, dt = \frac{3Va^2}{2r^2} \cos \theta \tag{2.20}$$

2.2 Orthogonal Functions and Expansions

Laplace Equation in Rectangular Coordinates

$$\nabla^2 \Phi(x, y, z) = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{2.21}$$

The variables may be separated by making the substitution

$$\Phi(x, y, z) = X(x)Y(y)Z(z) \tag{2.22}$$

whereby the Laplace equation reduces to

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \tag{2.23}$$

Here the first term depends only on x, the second only on y, and the third only on z. Equation 2.21 holds for arbitrary values of the independent coordinates only if each of the three terms must be separately constant:

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2, \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = -\beta^2, \qquad \frac{1}{Z}\frac{d^2Z}{dz^2} = \gamma^2 \tag{2.24}$$

where $\alpha^2 + \beta^2 = \gamma^2$.

Arbitrarily choosing $\alpha^2, \beta^2 > 0$ (then, $\gamma^2 > 0$), we obtain

$$\Phi(x, y, z) = e^{\pm i\alpha x} e^{\pm i\beta y} e^{\pm \gamma z} \tag{2.25}$$

Thus far there are no restrictions on α and β, but boundary conditions on the problem usually restrict them to a discrete set of values.
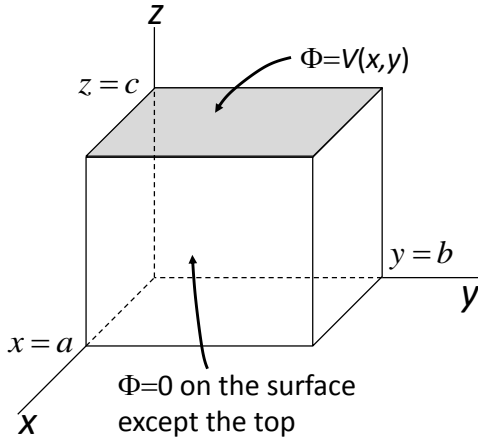
Rectangular cavity

Fig 2.6. Rectangular cavity with five sides at zero potential and the top ($z = c$) having specific potential $\Phi = V(x, y)$.

Consider a rectangular cavity shown in Fig. 2.6. The boundary condition is that $\Phi = V(x, y)$ on the top side while $\Phi = 0$ on the others.

(i) Using $\Phi = 0$ for $x = 0$, $y = 0$, $z = 0$,

$$\begin{aligned} X &= \sin \alpha x \\ Y &= \sin \beta y \\ Z &= \sinh \gamma z \end{aligned} \tag{2.26}$$

(ii) Using $\Phi = 0$ for $x = a$, $y = b$,

$$\alpha_n = \frac{n\pi}{a}, \qquad \beta_m = \frac{m\pi}{b}, \qquad \gamma_{nm} = \pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} \tag{2.27}$$

Then, the partial potential Φ_{nm} satisfying the boundary conditions of $\Phi = 0$

$$\Phi_{nm} = \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z) \tag{2.28}$$

(iii) The potential can be expanded in terms of Φ_{nm}

$$\Phi(x, y, z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z) \tag{2.29}$$

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where the coefficients A_{nm} are determined by the boundary condition $\Phi = V(x, y)$ at $z = c$:

$$V(x, y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c) \tag{2.30}$$

is a double Fourier series. Therefore, the Fourier coefficients A_{nm} can be obtained as

$$A_{nm} = \frac{4}{ab \sinh(\gamma_{nm} c)} \int_0^a dx \int_0^b dy \ V(x, y) \sin(\alpha_n x) \sin(\beta_m y) \tag{2.31}$$

Two infinite grounded metal plates

We consider two infinite grounded metal plates lying parallel to the yz plane. One end is closed off and its potential is maintained at V (see Fig. 2.7).
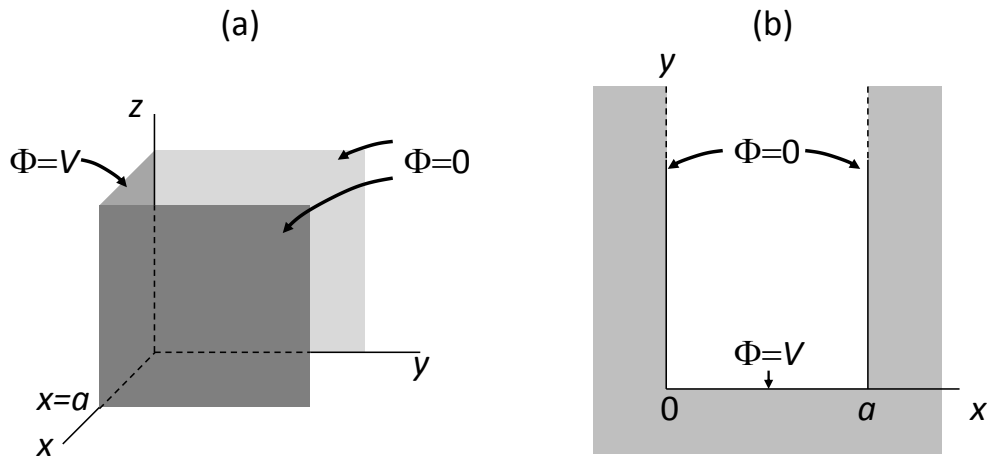

Fig 2.7. Two infinite grounded metal plates lie parallel to the yz plane, one at x=0, the other at x=a. The left end, y=0, is closed off with an infinite strip insulated from the two plates and maintained at $\Phi = V$.

This configuration is independent of z, so this is essentially a two-dimensional problem:

$$\nabla^2 \Phi(x, y) = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{2.32}$$

where separation of variable leads to $\Phi(x, y) = X(x)Y(y)$.

Then, the Laplace equation reduces to

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -\alpha^2, \qquad \frac{1}{Y}\frac{d^2 Y}{dy^2} = \alpha^2 \tag{2.33}$$

Since $\Phi = 0$ for $x = 0$ and a, $\Phi(x, y)$ can be written as

$$\Phi(x, y) = \sum_{n=1}^{\infty} A_n \, e^{-\frac{n\pi}{a}y} \sin\left(\frac{n\pi}{a}x\right)$$

(2.34)

From the last boundary condition, $\Phi = V$ for $y = 0$, $0 < x < a$,

$$A_n = \frac{2}{a}\int_0^a V \sin\left(\frac{n\pi}{a}x\right) dx = \frac{4V}{\pi n}\begin{cases} 1 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

Therefore,

$$\Phi(x, y) = \frac{4V}{\pi} \sum_{n=1,3,5,\dots} \frac{1}{n} \, e^{-\frac{n\pi}{a}y} \sin\left(\frac{n\pi}{a}x\right)$$
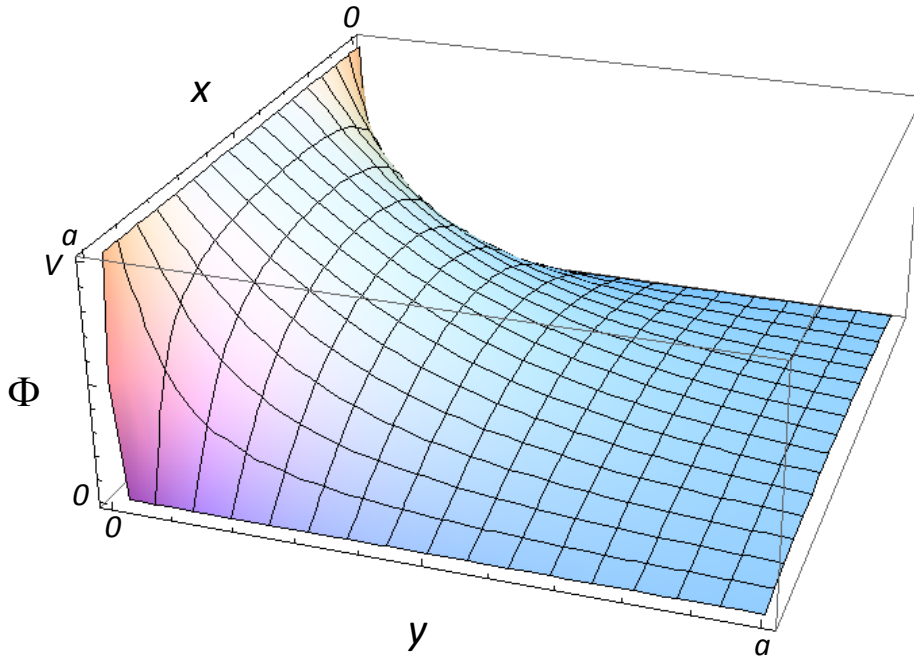
(2.35)

Fig 2.8. 3D plot of $\Phi(x, y)$

The infinite series in Eq. 2.35 can be summed up and its explicit form is

$$\Phi(x, y) = \frac{2V}{\pi} \tan^{-1}\left(\frac{\sin\frac{\pi}{a}x}{\sinh\frac{\pi}{a}y}\right)$$

(2.36)

9

Fields and Charge Densities in two-Dimensional Corners and Along Edges

We consider two infinite conducting planes forming a two-dimensional corner as shown in Fig. 2.9. The conducting planes maintain the potential at *V*, while the boundary conditions remote from the origin are not specified. We use polar coordinates for convenience.
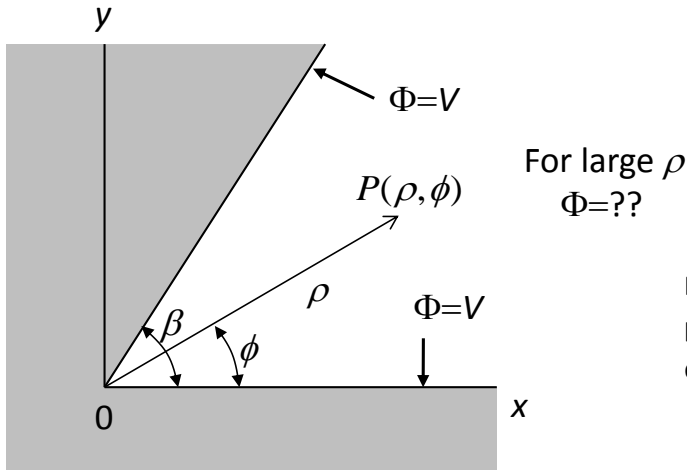

Fig 2.9. Intersection of two conducting planes defining a corner in two dimensions with opening angle β

The Laplace equation in polar coordinates is

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0 \tag{2.37}$$

Separating the variables by making the substitution

$$\Phi(\rho, \phi) = R(\rho)\Psi(\phi) \tag{2.38}$$

Then we obtain

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = -\frac{1}{\Psi}\frac{d^2\Psi}{d\phi^2} = \nu^2 \tag{2.39}$$

The solutions to Eq. 2.39 are

$$\begin{cases} R(\rho) = a\rho^\nu + b\rho^{-\nu} \\ \Psi(\phi) = A\cos(\nu\phi) + B\sin(\nu\phi) \end{cases} \quad \text{for } \nu \neq 0 \tag{2.40}$$

$$\begin{cases} R(\rho) = a_0 + b_0 \ln\rho \\ \Psi(\phi) = A_0 + B_0\phi \end{cases} \quad \text{for } \nu = 0 \tag{2.41}$$

The potential is finite at $\rho = 0$, therefore $b = b_0 = 0$. Since $\Phi = V$ for all ρ when $\phi = 0$ and β, $A = 0$, $B_0 = 0$, and $A_0 = V$. Furthermore, $\sin\nu\beta = 0$ requires

$$\nu = \frac{m\pi}{\beta}, \quad m = 1,2,\dots \tag{2.42}$$

Hence the general solution becomes

$$\Phi(\rho, \phi) = V + \sum_{m=1}^{\infty} a_m \rho^{\frac{m\pi}{\beta}} \sin\left(\frac{m\pi}{\beta}\phi\right) \tag{2.43}$$

where the coefficients a_m are determined by the remote boundary condition for large ρ. Our interest is in the electric fields and charge distribution around the corner. The potential near $\rho = 0$ is dominated by the first term in the series:

$$\Phi(\rho, \phi) \cong V + a_1 \rho^{\frac{\pi}{\beta}} \sin\left(\frac{\pi}{\beta}\phi\right) \tag{2.44}$$

The electric field components are

$$\begin{cases} E_\rho(\rho, \phi) = -\dfrac{\partial\Phi}{\partial\rho} \cong -\dfrac{\pi a_1}{\beta}\rho^{\frac{\pi}{\beta}-1}\sin\left(\dfrac{\pi}{\beta}\phi\right) \\[4mm] E_\phi(\rho, \phi) = -\dfrac{1}{\rho}\dfrac{\partial\Phi}{\partial\rho} \cong -\dfrac{\pi a_1}{\beta}\rho^{\frac{\pi}{\beta}-1}\cos\left(\dfrac{\pi}{\beta}\phi\right) \end{cases} \tag{2.45}$$

From the geometrical symmetry, we know that the surface-charge densities at $\phi = 0$ and β are equal:

$$\sigma(\rho) = \varepsilon_0 E_\phi(\rho, 0) \cong -\frac{\pi\varepsilon_0 a_1}{\beta}\rho^{\frac{\pi}{\beta}-1} \propto \rho^{\frac{\pi}{\beta}-1} \tag{2.46}$$

Figure 2.8 shows how the surface charge density varies as a function of ρ for several opening angles β.
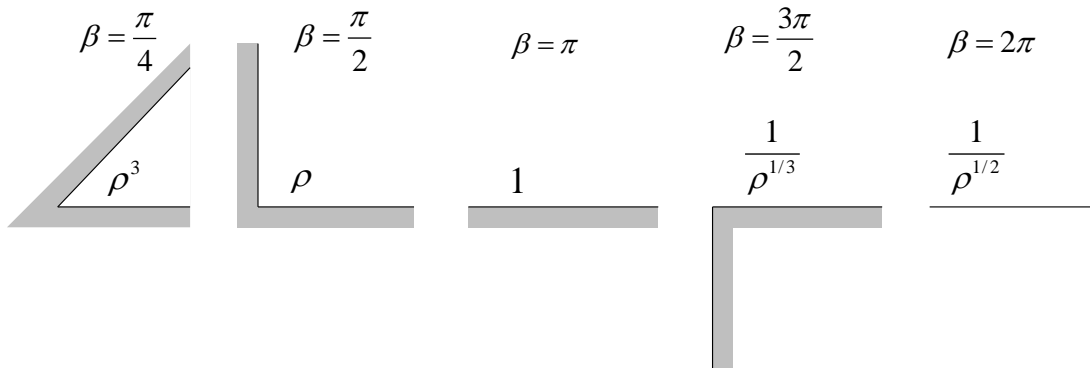

$$\beta = \frac{\pi}{4} \qquad \beta = \frac{\pi}{2} \qquad \beta = \pi \qquad \beta = \frac{3\pi}{2} \qquad \beta = 2\pi$$

$$\rho^3 \qquad\qquad \rho \qquad\qquad 1 \qquad\qquad \frac{1}{\rho^{1/3}} \qquad\qquad \frac{1}{\rho^{1/2}}$$

Fig 2.8. ρ dependent variation of the surface charge density for various corner angles