This document considers lines and circles in the complex plane and the inverse transformation $1/z$ acting on these subsets. It is shown that lines are mapped into lines or circles and, similarly, circles are mapped into lines of circles.

1 Equations of lines and circles

1.1 Lines

Let a line have Cartesian coordinates representation $ax + by + c = 0$, a, b, c are real constants. In complex plane this corresponds to a set of points z such that $x = Rez$ and $y = Imz$. Then $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/2i$. So, the equation is

$$a(z + \bar{z})/2 + b(z - \bar{z})/2i + c = 0.$$
$$z(a/2 + b/2i) + \bar{z}(a/2 - b/2i) + c = 0.$$

Denoting $\beta = a/2 + b/2i$, we see that the equation is

$$\beta z + \bar{\beta}\bar{z} + c = 0.$$

Reversing the calculation, we see that any equation $\beta z + \bar{\beta}\bar{z} + c = 0$ is an equation of a line $ax + by + c = 0$ where $a = \beta + \bar{\beta}$ and $b = \beta - \bar{\beta}$.

1.2 Circles

Consider a circle with radius r and center z_0. Then $|z - z_0| = r$, i.e., $(z - z_0)(\bar{z} - \bar{z_0}) = r^2$. We get

$$z\bar{z} - z_0\bar{z} - z\bar{z_0} = r^2 - |z_0|^2.$$

Moreover, any equation of the form

$$z\bar{z} - \beta z - \bar{\beta}\bar{z} = c,$$

for $c \geq -|\beta|^2$ is an equation of a circle with center at $\bar{\beta}$ and radius $(c + |\beta|^2)^{1/2}$.

2 Inverse Transformation of Circles and Lines

2.1 Lines not through the origin

Consider a line with equation $\beta z + \bar{\beta}\bar{z} + c = 0$, $c \neq 0$, a real number. Let $w = 1/z$, then $z = 1/w$ and the equation becomes:

$$\beta/w + \bar{\beta}\frac{1}{\bar{w}} + c = 0.$$
$$\beta\bar{w} + \bar{\beta}w + cw\bar{w} = 0.$$
$$(\beta/c)\bar{w} + (\bar{\beta}/c)w + w\bar{w} = 0.$$

This is an equation of a circle with center $-\bar{\beta}/c$.

2.2 Lines through the origin

A line through the origin has an equation $\beta z + \bar{\beta}\bar{z} = 0$. Apply transformation $w = 1/z$ as before to get

$$\beta/w + \bar{\beta}\frac{1}{\bar{w}} = 0.$$
$$\beta\bar{w} + \bar{\beta}w = 0.$$

This is again an equation of a line through the origin.

2.3 Circles not through the origin

Let
$$z\bar{z} - \bar{z}_0 z - z_0\bar{z} = r^2 - |z_0|^2,$$
$|z_0|^2 \neq r^2$. Let $w = 1/z$. Then
$$(1/w)(\bar{1/w}) - \bar{z}_0/w - z_0/\bar{w} = r^2 - |z_0|^2,$$
$$1 - \bar{z}_0\bar{w} - z_0 w = (r^2 - |z_0|^2)w\bar{w},$$
$$1/(r^2 - |z_0|^2) = (\bar{z}_0/(r^2 - |z_0|^2))\bar{w} + (z_0/(r^2 - |z_0|^2))w + w\bar{w}.$$

This is again an equation of a circle.

2.4 Circles through the origin

Let
$$z\bar{z} - \beta z - \bar{\beta}\bar{z} = 0,$$
let $w = 1/z$.

Then
$$(1/w)(\bar{1/w}) - \beta/w - \bar{\beta}/\bar{w} = 0,$$
$$1 - \beta\bar{w} - \bar{\beta}w = 0.$$

This is an equation of a line.

3 Direct calculation

Here, we achieve a similar result by a direct calculation.

Lemma 3.1. *A circle with radius r and center z_0 is mapped into a circle with radius $\frac{r}{|z_0|^2 - r^2}$ and center $\frac{\bar{z}_0}{(|z_0|^2 - r^2)}$ under the transform $1/z$ if $|z_0| \neq r$.*

Proof.
$$\begin{aligned}
|z - z_0| = r,\ w = 1/z \\
|1/w - z_0| = r \\
(1/w - z_0)(1/\bar{w} - \bar{z}_0) = r^2 \\
1/w\bar{w} - z_0/\bar{w} - \bar{z}_0/w + z_0\bar{z}_0 = r^2 \\
1 - z_0 w - \bar{z}_0\bar{w} + (|z_0|^2 - r^2)w\bar{w} = 0 \\
w\bar{w} - \frac{z_0}{(|z_0|^2 - r^2)}w - \frac{\bar{z}_0}{(|z_0|^2 - r^2)}\bar{w} + \frac{1}{(|z_0|^2 - r^2)} = 0 \\
\left(w - \frac{\bar{z}_0}{(|z_0|^2 - r^2)}\right)\left(\bar{w} - \frac{z_0}{(|z_0|^2 - r^2)}\right) = \frac{\bar{z}_0 z_0}{(|z_0|^2 - r^2)^2} - \frac{1}{(|z_0|^2 - r^2)} \\
\left(w - \frac{\bar{z}_0}{(|z_0|^2 - r^2)}\right)\left(\bar{w} - \frac{z_0}{(|z_0|^2 - r^2)}\right) = \frac{r^2}{(|z_0|^2 - r^2)^2}.
\end{aligned}$$

□

Lemma 3.2. *If $|z_0| = r$ then the circle with radius r and center z_0 is mapped into a line .. under transform $1/z$.*

Proof.

$$
\begin{aligned}
|z - z_0| = r,\ w = 1/z \\
|1/w - z_0| = r \\
(1/w - z_0)(1/\bar{w} - \bar{z}_0) = r^2 \\
1/w\bar{w} - z_0/\bar{w} - \bar{z}_0/w + z_0\bar{z}_0 = r^2 \\
1/w\bar{w} - z_0/\bar{w} - \bar{z}_0/w = 0 \\
1 - z_0 w - \bar{z}_0\bar{w} = 0 \\
1 = 2Re(z_0 w) \\
Re(z_0 w) = 1/2
\end{aligned}
$$

Thus $z_0 w$ has a parametric equation $z_0 w = 1/2 + it$, $t \in \mathbb{R}$, so $w = z_0/|z_0|^2(1/2 + it)$, $t \in \mathbb{R}$. □