GREEN'S FUNCTION :-

Consider a homogeneous differential equation of order n, say, $Ly = 0$ —— ①

Where L is the differential operator defined by

$$L = p_0(x)\frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{d}{dx}$$

$$+ p_n(x) \quad \text{——} ②$$

Where the functions $p_0(x), p_1(x), \ldots p_n(x)$ are continuous on $[a,b]$ and also $p_0(x) \neq 0$ on $[a,b]$ and the boundary conditions are

$$V_k(y) = 0 \; ; \; k = 1, 2, \ldots n \quad \text{——} ③$$

Where $V_k(y) = \alpha_k y(a) + \alpha_k' y'(a) + \cdots + \alpha_k^{n-1} y^{n-1}(a)$

$$+ \beta_k y(b) + \beta_k' y'(b) + \cdots + \beta_k^{n-1} y^{n-1}(b)$$

—— ④

Where the linear functions $V_1, V_2, \ldots V_n$ in $y(a), y'(a), \ldots y^{n-1}(a)$ & $y(b), y'(b), \ldots y^{n-1}(b)$ are L.I.

(Green's function of this DE ① to ④ exists only if trivial solution $y(x) = 0$ exists).

Suppose that homogeneous BVP given by ① to ④ has only trivial solution i.e. $y(x) \equiv 0$

then the Green's function of the BVP ① to ④ constructed for any t ; $a \le t \le b$ and which has the following properties.

1). In each of interval $[a,t]$ & $[t,b]$, the function $G(x,t)$ (Green's function) considered as a function of x, is a solution of equation ①

i.e. $\boxed{L(G)=0}$

2). $G(x,t)$ is continuous and has derivative w.r.t. 'x' upto order $(n-2)$ in $a \le x \le b$.

3). $(n-1)^{th}$ derivative of $G(x,t)$ w.r.t. 'x' at the point $x=t$ has a discontinuity of Ist kind.

The jump being equal to $-\frac{1}{p_0(t)}$

i.e. $$\boxed{\left(\frac{\partial^{n-1} G}{\partial x^{n-1}}\right)_{x=t^+} - \left(\frac{\partial^{n-1} G}{\partial x^{n-1}}\right)_{x=t^-} = \frac{-1}{p_0(t)}}$$

4). $G(x,t)$ satisfies B.C. ④

i.e. $V_k(G) = 0$; $k = 1, 2, \text{---} n.$

Theorem :- In the BVP ① to ④ has only trivial solution i.e. ($y(x) \equiv 0$) then the operator L has a unique Green's function.

Problem :- Find the Green's function of the BVP

$$y'' = 0 \quad \text{—①} \qquad ; \; y(0) = 0, \; y(l) = 0 \quad \text{—②}$$

Solution :- Firstly we check whether this problem has trivial solution or not.

$$y'' = 0$$

$$\Rightarrow y = Ax + B$$

$$y(0) = 0 \Rightarrow B = 0$$

$$\& \; y(l) = 0 \Rightarrow A = 0$$

$$\therefore \; y(x) \equiv 0$$

$\therefore$ The given BVP has a trivial solution.

Hence Green's function exists and unique.

Now Green's function has the following properties :-

1). In $[a, t]$; $[t, b]$.

Green's function should be solution of given BVP.

$\therefore$ Let

$$G(x,t) = \begin{cases} a_1 x + a_2 & ; \; 0 \le x \le t \\ b_1 x + b_2 & ; \; t \le x \le l \end{cases} \quad \text{—③}$$

& now we have to find the values of a_1, a_2, b_1, b_2.

2). $G(x,t)$ is continuous and $(n-2)$th derivative of $G(x,t)$

i.e. $G(x,t)$ is continuous at $x=\underline{t}$.

i.e. $$G(x,t)\Big/_{x=t^-} = G(x,t)\Big/_{x=t^+}$$

$$a_1 t + a_2 = b_1 t + b_2$$

$$\Rightarrow (b_1 - a_1)t + b_2 - a_2 = 0$$

$$\Rightarrow c_1 t + c_2 = 0 \quad \text{——— (4)}$$

where $c_i = b_i - a_i \; ; \; i = 1, 2.$

3). $(n-1)$th derivative of $G(x,t)$ has Ist kind of discontinuity.

$$G'(x,t) = \begin{cases} a_1 & ; \quad a \le x \le t \\ b_1 & ; \quad t \le x \le l \end{cases}$$

& $$\left(\frac{\partial G}{\partial x}\right)_{x=t^+} - \left(\frac{\partial G}{\partial x}\right)_{x=t^-} = -\frac{1}{p_0(t)} = -1$$

$$b_1 - a_1 = -1 \quad \text{——— (5)}$$

4). Green's function should satisfy the B.C.

i.e. $G(0)=0 \Rightarrow \boxed{a_2 = 0}$

& $G(l)=0 \Rightarrow b_1 l + b_2 = 0$ ——— (*)

As $c_1 t + c_2 = 0$

$\Rightarrow c_2 = t$

$\Rightarrow b_2 - a_2 = t$

$\Rightarrow b_2 - 0 = t \Rightarrow \boxed{b_2 = t}$

$\circledast \Rightarrow \quad b_1 l + t = 0$

$$\Rightarrow \boxed{b_1 = -\frac{t}{l}}$$

Now $\quad b_1 - a_1 = -1$

$$\Rightarrow -\frac{t}{l} - a_1 = -1 \quad \Rightarrow \quad a_1 = 1 - \frac{t}{l}$$

$$\Rightarrow \boxed{a_1 = \frac{l-t}{l}}$$

Put all these values in ③, we get

$$G(x,t) = \begin{cases} \frac{(l-t)}{l} x & ; \quad 0 \le x \le t \\ -\frac{t}{l} x + t & ; \quad t \le x \le l \end{cases}$$

i.e. $$G(x,t) = \begin{cases} \frac{(l-t)x}{l} & ; \quad 0 \le x \le t \\ \frac{(l-x)t}{l} & ; \quad t \le x \le l \end{cases}$$ Ans

V.V.I.

Remark :-

Green's function is symmetric

i.e. $\boxed{G(x,t) = G(t,x)}$