Complex analytic function:

Function $f(z)$ is analytic if it satisfy Cauchy-Reamann equation.

Condition: $f(z) = u + iv$

(necessary) $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$ & $\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$

Sufficent cond^n: $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ are continuous function of x & y.

examples: Polynomial, constant, exponential functions.

→ $\sin z$, $\cos z$

→ $\ln z$ (except $z=0$)

→ $\tan z$, $\sec z$ (except $z = (2n+1)\pi/2$)

→ $\cot z$, $\operatorname{cosec} z$ (except $z = n\pi$)

→ $z^{1/n}$ (except $z=0$)

Harmonic Function:

→ Function which satisfy laplace equation ($\nabla^2 A = 0$)

→ If $u+iv$ is an analytic, then u & v are conjugate Harmonic function.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \& \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Singularity of complex function:

Singular Point: Point at which function is not analytic

example: $f(z) = \frac{1}{z-1}$ the $z=1$ singular Point.

Isolated Singularity:

A point $z=z_0$ is said to be isolated singularity of $f(z)$ if:

1) $f(z)$ is not analytic at $z=z_0$
2) $f(z)$ is analytic in the neighbourhood of $z=z_0$

1) **Removable Singularity:**

If principal part of Laurent series expansion of $f(z)$ about $z=a$ contains no term.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

→ $\lim_{z \to z_0} f(z)$ is defined & $f(z_0)$ is not defined.

example: $\frac{\sin z}{z}$,

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

ie. Removable singularity.

2) **Non-essential singularity or Pole:**

$$f(z) = \frac{P_1(z)}{Q_1(z)}$$

order of Pole = degree of function $Q(z)$

For Poles: Determinant $(D^T) = 0$

example: $\frac{\sin z}{z^3} \Rightarrow \frac{\sin z}{z} \frac{1}{z^2}$ Pole of order 2.

3) **Essential singularity:**

If $z=z_0$ is neither pole nor removable, then it is known as essential singularity.

3) Essential singularity:

If $z=z_0$ is neither pole nor removable, then it is known as essential singularity.

example: $f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \dots$

has essential singularity at $z=0$

Non-isolated singularity: (12)

A point $z=z_0$ is said to be non-isolated singularity of $f(z)$ if:

a) $f(z)$ is not analytic at $z=z_0$

b) there exists a neighbourhood of $z=z_0$, containing other singularities of $f(z)$

example: $f(z) = \frac{1}{\sin(\pi/z)}$

Ques 1 The function $\frac{z}{\sin \pi z^2}$ of a complex variable z has

a) a simple pole at 0 & poles of order 2 at $z = \pm\sqrt{n}$ for $n = 1, 2, 3, \dots$

b) a simple pole at 0 & poles of order 2 at $z = \pm\sqrt{n}$ & $z = \pm i\sqrt{n}$ for $n = 1, 2, 3, \dots$

c) Poles of order 2 at $z = \pm\sqrt{n}$; $n = 0, 1, 2, \dots$

d) Poles of order 2 at $z = \pm n$; $n = 0, 1, 2, 3, \dots$

Sol^n:- $f(z) = \frac{z}{\sin(\pi z^2)}$

for Poles: $D^r = 0$

$\sin(\pi z^2) = 0 \Rightarrow \pi z^2 = n\pi$; $n = 0, \pm 1, \pm 2$

$z^2 = \pm n$; $n = 0, 1,$

$z = \pm\sqrt{n}, \pm i\sqrt{n} \rightarrow$ Pole of order 2. $\boxed{B}$

Cauchy Residue Formula:

If $f(z)$ is single valued & analytic in a simple closed curve, except at some finite number of singular points then

$$\oint_C f(z)\,dz = 2\pi i \times (\text{sum of Residues at Poles within 'C'})$$

Cauchy Theorem: If $f(z)$ is an analytic function within or on the contour 'C'

$$\oint f(z)\,dz = 0$$

Residue of complex number:

→ **Residue at simple Pole:**

① If $f(z) = \dfrac{P(z)}{Q(z)}$ & $Q(z) \neq 0$

then Residue at $z = z_0$ is:

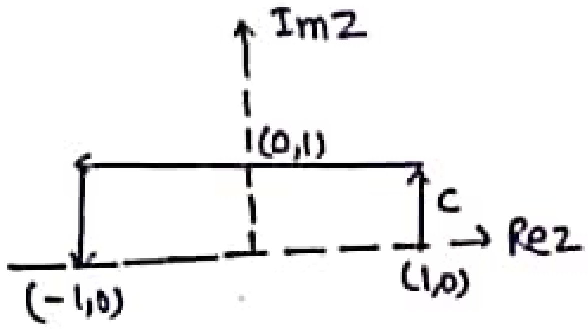
$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \frac{P(z)}{Q'(z)}}$$

→ **Residue at a pole of order n:**

②
$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^m f(z)\right]\right]}$$

③ Put $z+a=t$ & expend f^n. Res = coeff. of $\frac{1}{t}$

Que1 The value of the integral $\int_C dz\, z^2 e^2$, where C is an oper contour in the complex z-plane as shown in figure below:



a) $\frac{5}{e}+e$ b) $e-\frac{5}{e}$ c) $\frac{5}{e}-e$ d) $-\frac{5}{e}-e$

Sol^n: $f(z) = z^2e^z$ → analytic fxn

By cauchy theorem:

$$\oint z^2e^z dz = 0$$

$$\int_C z^2e^z dz + \int_{-1}^{1} z^2e^z dz = 0$$

$$\int_C z^2e^z dz = -\int_{-1}^{1} z^2e^x dz$$

$$= -\left[z^2e^z - 2ze^z + 2e^z\right]_{-1}^{1}$$

$$= -\left[e - 2e + 2e - e^{-1} - 2e^{-1} - 2e^{-1}\right]$$

$$= 5e^{-1} - e$$

$$= \frac{5}{e} - e \qquad \boxed{C}$$

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Que: The principle value of the Real integral

$$I=\int_{-3}^{3}\frac{dx}{x^2+3x+2} \text{ is:}$$

a) $3\pi/2$ b) $\ln(2/5)$ c) ∞ d) 0

Solⁿ:-
$$I=\int_{-3}^{3}\frac{dx}{x^2+3x+2}=\int_{-3}^{3}\frac{dx}{(x+1)(x+2)}$$

simple poles at $x=-1,-2$

its principle value can be calculated by using contour integration.

$$I=\oint_C \frac{dz}{(z+1)(z+2)}$$

Poles lies on Real axis: $z=-1,-2$

$$I=2\pi i\times\left[\underset{z=-1}{\text{Res}}\,f(z)+\underset{z=-2}{\text{Res}}\,f(z)\right]$$

$$\underset{z=-1}{\text{Res}}\,f(z)=\lim\frac{p(z)}{Q'(z)}\bigg|_{z=-1}=\frac{1}{2z+3}\bigg|_{z=-1}.$$

$$\text{Res}\,f(z)\Big|_{z=-1}=1$$

$$\text{Res}\,f(z)\Big|_{z=-2}=\frac{1}{2z+3}\bigg|_{z=-2}=-1$$

$$I=2\pi i\times[1-1]=0 \qquad \boxed{D}$$

Que 3 The value of the integral

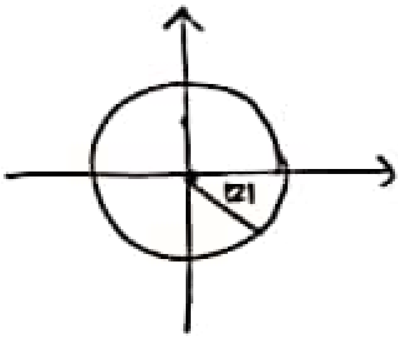
$$\int_C \frac{z^3\,dz}{z^5-5z+6} \quad \text{\&} \quad 2|z|-5=0$$

a) $-16\pi i$ b) $16\pi i$ c) $8\pi i$ d) $2\pi i$

Solⁿ:- $f(z)=\dfrac{z^3}{z^2-5z+6}$

$C:\ 2|z|-5=0$

$|z|=5/2$ ← Radius



For Poles: DT=0

$z^2-5z+6=0$

$z^2-3z-2z+6=0$

$z=3,2$ ← simple Poles

$z=2$ inside the circle,

$$I=2\pi i \times \operatorname{Res} f(z)\Big|_{z=2}$$

$$I=2\pi i \times \frac{z^3}{2z-5}\Bigg|_{z=2} \Rightarrow 2\pi i \times \frac{8}{-1}$$

$$I=-16\pi i \qquad \boxed{A}$$

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Ques 4 If c is the contour defined by $|z| = 1/2$, the value of integral $\oint_c \frac{dz}{\sin^2 z}$ is: (47)

a) ∞ b) $2\pi i$ c) 0 d) πi

Solⁿ:- $f(z) = \frac{1}{\sin^2 z}$ & $|z| = 1/2$

For Poles: $\sin^2 z = 0$

Pole of order 2 at $z = n\pi$

only $z = 0$ lies inside $|z| = 1/2$

$$f(z) = \frac{1}{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)^2}$$

$$= \frac{1}{z^2}\left[1 - \frac{z^2}{3!} + \frac{z^5}{5!} - \cdots\right]^{-2}$$

coefficient of $(1/z) = 0$

$$\text{Res } f(z)\Big|_{z=0} = 0$$

$I = 2\pi i \times \text{Res.} \Rightarrow 0$ $\boxed{c}$

Ques 5 The value of the contour intergal $\frac{1}{2\pi i}\oint \frac{e^{4z}-1}{\cosh z-2\sinh z}\,dz$ around the unit circle C transversed in the anticlockwise direction is:

(c)

a) 0 b) 2 c) $-\pi/\sqrt{3}$ d) $-\tanh(1/2)$

Soln:- $$f(z) = \frac{e^{4z}-1}{\cosh z - 2\sinh z} = \frac{e^{4z}-1}{\frac{e^{z}+e^{-z}}{2} - 2\,\frac{(e^{z}-e^{-z})}{2}}$$

$$= \frac{2(e^{4z}-1)}{3e^{-z}+e^{z}}$$

For Poles: $D^r = 0$

$$3e^{-z} = -e^{+z} \Rightarrow 3 = \frac{e^{z}}{e^{-z}}$$

$$3\bar{e}^{z}+e^{z}$$

For Poles: $D^T = 0$

$$3\bar{e}^{z} = -e^{+z} \Rightarrow 3 = \frac{e^{z}}{e^{-z}}$$

$$3 = e^{2z}$$

$$\Rightarrow \ln 3 = 2z$$

$$z_0 = \frac{\ln 3}{2} \rightarrow \text{simple Pole.}$$

(lies within circle)

$$\text{Res. } f(z)\Big|_{z=z_0} = \frac{P(z)}{Q'(z)} = \frac{2(e^{4z}-1)}{-3\bar{e}^{z}+e^{z}}\Bigg|_{z=z_0}$$

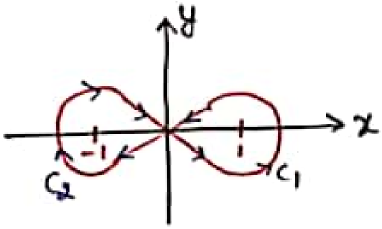
$$= \frac{2\left[e^{\frac{4}{2}\ln 3}-1\right]}{-3\,e^{\frac{-1}{2}\ln 3} - e^{+\frac{1}{2}\ln 3}}$$

$$= \frac{2\left[3^{2}-1\right]}{3\cdot\frac{1}{\sqrt{3}}-\sqrt{3}} = \frac{-8}{\sqrt{3}}$$

now, $I = \frac{1}{2\pi i}\oint f(z)\,dz$

$$= \frac{1}{2\pi i} \times 2\pi i \times \left(\frac{-8}{\sqrt{3}}\right) = \frac{-8}{\sqrt{3}} \quad \boxed{C}$$

Que6 The integral $\oint_{\Gamma} \frac{z e^{i\pi z/2}}{z^2-1}\,dz$ along the closed contour Γ shown in the figure is:

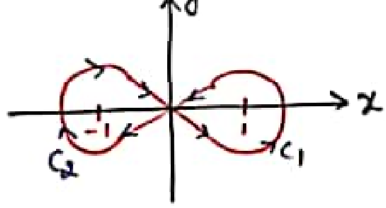


a) 0 b) 2π

c) -2π d) $4\pi i$

Solⁿ:- $f(z) = \frac{z e^{i\pi z/2}}{z^2-1}$

For Poles: $z^2-1=0 \Rightarrow z = \pm 1$ → simple Pole



a) 0 b) 2π

c) -2π d) $4\pi i$

Solⁿ:- $f(z) = \dfrac{z e^{i\pi z/2}}{z^2 - 1}$

For Poles : $z^2 - 1 = 0 \Rightarrow z = \pm 1$, simple Pole

$$\operatorname{Res} f(z)\Big|_{z=1} = \frac{P(z)}{Q'(z)}\Big|_{z=1} = \frac{1 \cdot e^{i\pi/2}}{2\cdot(1)} = \frac{1}{2} e^{i\pi/2}$$

$$\operatorname{Res} f(z)\Big|_{z=-1} = \frac{z e^{i\pi z/2}}{2z}\Big|_{z=-1} = +\frac{e^{-i\pi/2}}{2}$$

$$I = 2\pi i \times \left[\frac{1}{2} e^{i\pi/2} + \frac{1}{2} e^{-i\pi/2}\right]$$

$$= \pi i \times \cancel{\sin}\ 2i(\sin \pi/2)$$

$$= -2\pi \qquad \boxed{C}$$

5

Improper Integral:

between limits $-\infty$ to $+\infty$

Theorem I:

If $f(x)$ contains only Polynomial terms.

then $f(z) \equiv f(x)$

→ Find singular Points

→ check points lie in upper Half.

A) If singular Points lie on ~~imagin~~ imaginary axis.

$$\int_{-\infty}^{\infty} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= 2\pi i \times [\Sigma \text{Res.}] - \underbrace{\left(\lim_{z\to\infty} z f(z)\right) \times \pi i}_{\uparrow\, 2\text{nd}}$$

B) If singular Point lie on Real axis.

$$\int_{-\infty}^{\infty} f(z)\,dz = \pi i \times [\Sigma \text{Res.}] - \underbrace{\pi i \left[\lim_{z\to\infty} z f(z)\right]}_{\uparrow\, 2\text{nd}}$$

A) If singular Points lie on ~~imagin~~ imaginary axis.

$$\int_{-\infty}^{\infty} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= 2\pi i \times \left[\Sigma \text{Res.}\right] - \underbrace{\left(\lim_{z\to\infty} z f(z)\right) \times \pi i}_{\uparrow 2nd}$$

B) If singular Point lie on Real axis.

$$\int_{-\infty}^{\infty} f(z)\,dz = \pi i \times \left[\Sigma \text{Res.}\right] - \underbrace{\pi i \left[\lim_{z\to\infty} z f(z)\right]}_{\uparrow 2nd}$$

Theorem II:

→ If $f(x)$ contain sine, cosine function along with Polynomial fxn.

then, Rule is same except 2nd term $(\underline{\quad \text{ind.} \quad})$ which is 0 in this case.

Que 1 The value of the integral

$$\int_{-\infty}^{\infty} \frac{1}{t^2 - R^2} \cos\left(\frac{\pi t}{2R}\right) dt$$

a) $-\frac{2\pi}{R}$ b) $-\frac{\pi}{R}$ c) $\frac{\pi}{R}$ d) $\frac{2\pi}{R}$

Soln:

$$\int_{-\infty}^{\infty} f(t)\,dt \Rightarrow f(t) = \frac{1}{t^2 - R^2} \cos\left(\frac{\pi t}{2R}\right)$$

$$f(t) = \frac{1}{t^2 - R^2} \text{ Real Part} \left(\exp\left(i \frac{\pi t}{2R}\right)\right)$$

$$f(z) = \frac{1}{z^2 - R^2} \exp i \left(\frac{\pi z}{2R}\right)$$

$$f(z) = \frac{1}{z^2-R^2} \exp i\left(\frac{\pi z}{2R}\right)$$

$z^2 - R^2 = 0 \Rightarrow z = \pm R \rightarrow$ simple Pole.

lies on Real axis.

$$\operatorname{Res} f(z)\Big|_{z=R} = \frac{1}{2R} \exp i(\pi/2)$$

$$\operatorname{Res} f(z)\Big|_{z=-R} = -\frac{1}{2R} \exp i(-\pi/2)$$

$e^{i\pi/2} = \sin\pi/2$

$e^{-i\pi/2} = -\sin\pi/2$

$(1-(-1)) = 2$

I = Real Part(πi x sum of Residue)

$$= \text{Real}\left(\frac{\pi i}{2R}\left[e^{i\pi/2} - \frac{1}{2R} e^{-i\pi/2}\right] = \frac{2\pi}{2R} i \quad \boxed{B}\right.$$

$(I = \pi/R)$

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Ques The principle value of the integral

$$\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^3} dx \text{ is:}$$

a) -2π b) $-\pi$ c) π d) 2π

Soln:- $I = \int_{-\infty}^{\infty} \frac{\sin 2x}{x^3} dx$ = img. Part of $\int_{-\infty}^{\infty} \frac{e^{i2x}}{x^3} dx$

let $f(z) = \frac{e^{i2z}}{z^3}$

$z=0$ is Pole of order 3 $\rightarrow$ lies on Real axis

$$\operatorname{Res} f(z)\Big|_{z=0} = \pi i \left[\frac{1}{(3-1)!} \frac{d^2}{dz^2}\left[(z-0)^3 \cdot \frac{e^{i2z}}{z^3}\right]\right]_{z=0}$$

$$= \frac{1}{2!} \frac{d^2}{dz^2} (e^{i2z})$$

$$= \frac{1}{2} \times (4i^2) \times e^0 \Rightarrow -2$$

$z=0$ is Pole of order 3 $\rightarrow$ lies on Real axis

$$\operatorname{Res} f(z)\Big|_{z=0} = \pi i \left[\frac{1}{(3-1)!}\frac{d^2}{dz^2}\left[(z-0)^3 \cdot \frac{e^{i2z}}{z^3}\right]\right]_{z=0}$$

$$= \frac{1}{2!}\frac{d^2}{dz^2}\left(e^{i2z}\right)$$

$$= \frac{1}{2} \times (4i^2) \times e^0 \Rightarrow -2$$

$$I = \pi i \times \operatorname{Res} f(z)\Big|_{z=0}$$

$$= \pi i \times (-2) = -2\pi i$$

required integral = img. Part of $(-2\pi i)$

$$= -2\pi \qquad \boxed{A}$$

Que 3 Given that the integral

$$\int_0^\infty \frac{dx}{y^2+x^2} = \frac{\pi}{2y}$$, the value of $$\int_0^\infty \frac{dx}{(y^2+x^2)^2}$$ is: (53)

a) $\frac{\pi}{y^3}$ b) $\frac{\pi}{4y^3}$ c) $\frac{\pi}{8y^3}$ d) $\frac{\pi}{2y^3}$

Solⁿ:- $$\int_0^\infty \frac{dx}{(y^2+x^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(y^2+x^2)}$$

For Poles: $(y^2+z^2)^2 = 0$

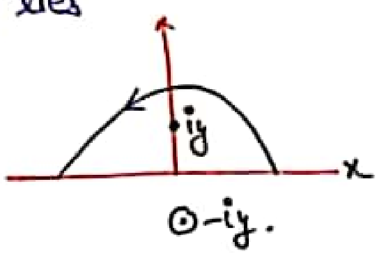
$z = \pm iy$ Pole of order 2

$\uparrow$

lies on imaginary axis.

on upper half $z = +iy$ lies

$$I = \frac{1}{2}\left(2\pi i \times \operatorname{Res} f(z)\Big|_{z=iy}\right)$$



Que 3 Given that the integral

$$\int_0^{\infty} \frac{dx}{y^2+x^2} = \frac{\pi}{2y} \text{ , the value of } \int_{-0}^{\infty} \frac{dx}{(y^2+x^2)^2} \text{ is:}$$

a) $\frac{\pi}{y^3}$ b) $\frac{\pi}{4y^3}$ c) $\frac{\pi}{8y^3}$ d) $\frac{\pi}{2y^3}$

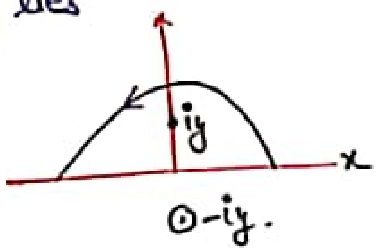
Solⁿ:-
$$\int_0^{\infty} \frac{dx}{(y^2+x^2)^2}\cancel{dx} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(y^2+x^2)}$$

For Poles: $(y^2+z^2)^2 = 0$

$z = \pm iy$ Pole of order 2

↑ lies on imaginary axis.

on upper half $z = +iy$ lies



$$I = \frac{1}{2}\left(2\pi i \times \operatorname{Res} f(z)\Big|_{z=iy}\right)$$

$$\operatorname{Res} f(z)\Big|_{z=iy} = \frac{1}{(2-1)!}\frac{d}{dz}\left[(z-iy)^2 \cdot \frac{1}{(z^2+y^2)^2}\right]\Bigg|_{z=iy}$$

$$= \frac{d}{dz}\left[(z-iy)^2 \cdot \frac{1}{(z+iy)^2(z-iy)^2}\right]\Bigg|_{z=iy}$$

$$\operatorname{Res} f(z)\Big|_{z=iy} = \frac{d}{dz}\left[\frac{1}{(z+iy)^2}\right]\Bigg|_{z=iy}$$

$$= \frac{-2}{(z+iy)^3}\Bigg|_{z=iy} = \frac{-2}{(2iy)^3}$$

$$= \frac{-2}{-8iy^3} = \frac{1}{4iy^3}$$

$$\text{Res } f(z)\Big|_{z=iy} = \frac{d}{dz}\left[\frac{1}{(z+iy)^2}\right]\Bigg|_{z=iy}$$

$$= \frac{-2}{(z+iy)^3}\Bigg|_{z=iy} = \frac{-2}{(2iy)^3}$$

$$= \frac{-2}{-8iy^3} = \frac{1}{4iy^3}$$

$$I = \pi i \times \frac{1}{4iy^3} = \frac{\pi}{4y^3} \qquad \boxed{B}$$

Ques The value of the integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$ is:

a) $\frac{\pi}{\sqrt{2}}$ b) $\frac{\pi}{2}$ c) $\sqrt{2}\pi$ d) 2π

Solⁿ: $f(z) = \frac{1}{1+z^4}$

For Poles: $z^4+1=0$

$$z^4 = -1 = e^{i(2n+1)\pi}$$

$$z = e^{i(2n+1)\pi/4} \quad n=0,1,2,\ldots$$

$$z = \underbrace{e^{i\pi/4},\ e^{i3\pi/4}}_{}, \ e^{i5\pi/4}, \ldots$$

↑ lies inside upper half. (0 to 2π)

$$\text{Res } f(z)\Big|_{z=e^{i\pi/4}} = \frac{P(z)}{Q'(z)}\Bigg|_{z_0} = \frac{1}{4z^3}\Bigg|_{z_0} = \frac{1}{4e^{i3\pi/4}}$$

$$\text{Res } f(z)\Big|_{z=e^{i3\pi/4}} = \frac{1}{4\, z^{i9\pi/4}}$$

$$I = 2\pi i \times \frac{1}{4}\left[\frac{1}{e^{i3\pi/4}} + \frac{1}{e^{i9\pi/4}}\right]$$

$$= \frac{2\pi i}{4}\left[e^{-3i\pi/4} + e^{-i9\pi/4}\right]$$

$$= \frac{\pi i}{2}\left[\frac{-1-i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}}\right]$$

For Poles: $z^4+1=0$

$$z^4 = -1 = e^{i(2n+1)\pi}$$

$$z = e^{i(2n+1)\pi/4}, \quad n=0,1,2,---$$

$$z = \underbrace{e^{i\pi/4}, e^{i3\pi/4}}_{\text{lies inside upper half. (0 to } 2\pi)}, e^{i5\pi/4}, ----$$

$$\text{Res } f(z)\Big|_{z=e^{i\pi/4}} = \frac{P(z)}{Q'(z)}\Big|_{z_0} = \frac{1}{4z^3}\Big|_{z_0} = \frac{1}{4e^{i3\pi/4}}$$

$$\text{Res } f(z)\Big|_{z=e^{i3\pi/4}} = \frac{1}{4z^{i9\pi/4}}$$

$$I = 2\pi i \times \frac{1}{4}\left[\frac{1}{e^{i3\pi/4}} + \frac{1}{e^{i9\pi/4}}\right]$$

$$= \frac{2\pi i}{4}\left[e^{-3i\pi/4} + e^{-i9\pi/4}\right]$$

$$= \frac{\pi i}{2}\left[\frac{-1-i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}}\right]$$

$$= \frac{2\pi}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}} \qquad \boxed{A}$$

using: $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

Note: Principal Branch of multivalued function is always analytic.

Characteristic of Branch Point:

→ at Branch Point multivalued function $f(z)$ is not analytic

→ At branch point, the values of branches of $f(z)$ are equal.