TOPIC 6

Infinite series 1: Geometric and telescoping series

Main ideas.

- Convergence and divergence: general definitions and intuitions
- Geometric series: $\sum_{k=0}^{\infty} r^k$
- Telescoping series $\sum_{k=*}^{\infty} \frac{1}{\text{quadratic}}$

Exercises.

Exercise 6.1. For each of the series below, please

- Write out the first few partial sums S_1, S_2, S_3
- Write out a general formula for S_n
- Determine if the series converges. If the series is convergent, to what does it converge? In either case, explain your reasoning.

(1) $\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{2^k}{3^k}$

(2) $\displaystyle\sum_{k=0}^{\infty} \frac{5^k}{4^k}$

(3) $\displaystyle\sum_{k=0}^{\infty} \frac{2}{4^k}$

(4) $\displaystyle\sum_{k=0}^{\infty} \left(\frac{1}{3^k} + \frac{5}{6^k}\right)$

(5) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+2k}$

(6) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k^2-4}$

(7) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^2+k}$

Solution:

(1) We have

$$S_0 = 1$$

$$S_1 = 1 - \frac{2}{3}$$

$$S_2 = 1 - \frac{2}{3} + \frac{4}{9}$$

$$S_3 = 1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27}$$

The general formula for S_n is

$$S_n = \sum_{k=0}^{n} (-1)^k \frac{2^k}{3^k} = \sum_{k=0}^{n} \left(-\frac{2}{3}\right)^k = \frac{1-\left(-\frac{2}{3}\right)^{n+1}}{1 - -\frac{2}{3}} = \frac{3}{5}\left[1 - \left(-\frac{2}{3}\right)^{n+1}\right]$$

As $n \to$ large we have $\left(-\frac{2}{3}\right)^{n+1} \to 0$ and thus the series is convergent, with

$$\sum_{k=0}^{\infty} (-1)^k \frac{2^k}{3^k} = \frac{3}{5}$$

(2) We compute

$$S_0 = 1$$
$$S_1 = 1 + \frac{5}{4}$$
$$S_2 = 1 + \frac{5}{4} + \frac{25}{16}$$

In general we have

$$S_n = \sum_{k=0}^{n} \frac{5^k}{4^k} = \frac{1-\left(\frac{5}{4}\right)^{n+1}}{1-\frac{5}{4}} = -4\left[1 - \left(\frac{5}{4}\right)^{n+1}\right]$$

Since $\left(\frac{5}{4}\right)^{n+1}$ gets large as n gets large, the series does not converge.

(3) We compute

$$S_0 = 2$$
$$S_1 = 2 + \frac{2}{4}$$
$$S_2 = 2 + \frac{2}{4} + \frac{2}{16}$$

The general formula for the partial sums is

$$S_n = \sum_{k=0}^{n} \frac{2}{4^k} = 2\sum_{k=0}^{n} \left(\frac{1}{4}\right)^k = 2\frac{1-\left(\frac{1}{4}\right)^{n+1}}{1-\frac{1}{4}} = \frac{8}{3}\left[1 - \left(\frac{1}{4}\right)^{n+1}\right].$$

Since $\left(\frac{1}{4}\right)^{n+1}$ gets small as n gets large we see that the series converges and

$$\sum_{k=0}^{\infty} \frac{2}{4^k} = \frac{8}{3}.$$

(5) We first do a partial fraction decomposition, writing

$$\frac{1}{k^2+2k} = \frac{1}{k(k+2)} = \frac{1}{2}\left[\frac{1}{k} - \frac{1}{k+2}\right].$$

Thus the partial sum S_n is given by

$$S_n = \frac{1}{2}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right.$$
$$\left. - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n+3}\right].$$

Many terms, and we see that

$$S_n = \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right].$$

As $n \to \infty$ we have $S_n \to \frac{1}{2}\left[1 + \frac{1}{2}\right]$; thus the series converges as

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + 2k} = \frac{3}{4}.$$

(6) A partial fraction decomposition gives us

$$\frac{1}{k^2 - 4} = \frac{1}{(k-2)(k+2)} = \frac{1}{4}\left[\frac{1}{k-2} - \frac{1}{k+2}\right].$$

Thus the partial sums are

$$\begin{aligned} S_n = \frac{1}{4}\Bigg[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-2} \\ -\frac{1}{5} - \frac{1}{6} - \cdots - \frac{1}{n+2}\Bigg]. \end{aligned}$$

There are 8 terms which do not cancel and thus

$$S_n = \frac{1}{4}\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{n-1} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}\right].$$

We can see that the sequence of partial sums converges and hence

$$\sum_{k=3}^{\infty} \frac{1}{k^2 - 4} = \frac{1}{4}\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right].$$

(7) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^2 + k}$ Clearly a factor of k cancels and the series is equivalent to

$$\sum_{k=1}^{\infty} \frac{1}{k+1}.$$

We look at the partial sums and use the same grouping as in class:

$$S_n = \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>\frac{1}{2}} + \cdots + \frac{1}{n+1}.$$

In particular, we notice that $S_{2^k - 1}$ ends with $\cdots + \frac{1}{2^k}$ and therefore

$$S_{2^k-1} = \underbrace{\frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{k \text{ times}} > \frac{1}{2}k.$$

Since $\frac{1}{2}k$ grows without bound, we see that the sequence of partial sums grows without bound and the series diverges.

Note: There are more problems on the next page!

Exercise 6.2. Write each of the following series in terms "standard" geometric series. Then determine whether it converges or not.

Example: Suppose we are given the series $\sum_{k=2}^{\infty} \frac{2^{2k+1}}{3^k}$. First we re-write as

$$\sum_{k=2}^{\infty} \frac{2^{2k+1}}{3^k} = \sum_{k=2}^{\infty} \frac{2 \cdot 4^k}{3^k} = 2\sum_{k=2}^{\infty} \left(\frac{4}{3}\right)^k = 2\sum_{k=2}^{\infty} \left(\frac{4}{3}\right)^{k-2} \left(\frac{4}{3}\right)^2 = \frac{32}{9} \sum_{k=2}^{\infty} \left(\frac{4}{3}\right)^{k-2}.$$

Next we define a new counter $l = k - 2$. Notice that when $k = 2$ our new counter has $l = 0$. Thus we have

$$\sum_{k=2}^{\infty} \frac{2^{2k+1}}{3^k} = \frac{32}{9} \sum_{l=0}^{\infty} \left(\frac{4}{3}\right)^l.$$

The series does not converge because the ratio $\frac{4}{3}$ is greater than 1.

(1) $\sum_{n=2}^{\infty} \frac{3^n}{4^n}$

(2) $\sum_{n=3}^{\infty} (-1)^n \frac{2}{3^n}$

(3) $\sum_{n=1}^{\infty} \frac{2}{3^{n+2}}$

(4) $\sum_{n=3}^{\infty} (-1)^n \frac{3^n}{2 \cdot 4^n}$

(5) $\sum_{n=0}^{\infty} \left(\frac{1}{3^{n-1}} - \frac{2}{9^n}\right)$

(6) $\sum_{n=-1}^{\infty} \frac{5^{n-3}}{6^{2n-1}}$

Solution:

(1) We write the series as

$$\sum_{n=2}^{\infty} \frac{3^n}{4^n} = \sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n.$$

Already we see that the series converges, as the geometric ratio $\frac{3}{4}$ is less than 1.

We now re-index, setting $k = n - 2$ and obtaining

$$\sum_{n=2}^{\infty} \frac{3^n}{4^n} = \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^{k+2} = \left(\frac{3}{4}\right)^2 \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k.$$

Optional: At this point we can easily determine the precise value that the series converges to; we find

$$\sum_{n=2}^{\infty} \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^2 \frac{1}{1 - \frac{3}{4}}$$

(2) $\sum_{n=3}^{\infty}(-1)^n \frac{2}{3^n}$ We may write the series as

$$\sum_{n=3}^{\infty}(-1)^n \frac{2}{3^n} = 2\sum_{n=3}^{\infty}\left(\frac{-1}{3}\right)^n.$$

Since $\left|\frac{-1}{3}\right| < 1$ the series converges.
Re-indexing, we set $l = n - 2$ and obtain

$$\sum_{n=3}^{\infty}(-1)^n \frac{2}{3^n} = 2\left(\frac{-1}{3}\right)^2 \sum_{l=0}^{\infty}\left(\frac{-1}{3}\right)^l.$$

Optional: At this point we can easily determine the precise value that the series converges to; we find

$$\sum_{n=3}^{\infty}(-1)^n \frac{2}{3^n} = 2\left(\frac{-1}{3}\right)^2 \frac{1}{1+\frac{1}{3}} = \frac{2}{3}\cdot\frac{3}{4} = \frac{1}{2}.$$

(3) $\sum_{n=-1}^{\infty}\frac{5^{n-3}}{6^{2n-1}}$ We write the sum as

$$\sum_{n=-1}^{\infty}\frac{5^{n-3}}{6^{2n-1}} = \frac{5^{-3}}{6^{-1}}\sum_{n=-1}^{\infty}\left(\frac{5}{36}\right)^n$$

from which we see that the series converges.
Re-indexing with $k = n + 1$ we find

$$\sum_{n=-1}^{\infty}\frac{5^{n-3}}{6^{2n-1}} = \frac{5^{-3}}{6^{-1}}\sum_{k=0}^{\infty}\left(\frac{5}{36}\right)^{k-1} = \frac{6^3}{5^4}\sum_{k=0}^{\infty}\left(\frac{5}{36}\right)^k.$$

Optional: Using the re-indexed expression for the sum we find

$$\sum_{n=-1}^{\infty}\frac{5^{n-3}}{6^{2n-1}} = \frac{6^3}{5^4}\frac{1}{1-\frac{5}{36}}$$

Exercise 6.3. (Challenge) Recall that the geometric sum formula gives us a nice expression for the quantity

$$1 + x + x^2 + \cdots + x^n.$$

Use this to find a nice formula for the quantity

$$1 + 2x + 3x^n + \cdots + nx^{n-1}.$$

Use your formula to analyze the convergence of the series

$$\sum_{k=1}^{\infty} k\left(\frac{1}{3}\right)^{n-1}.$$