Important

(23) For positive term sequence $\langle a_n \rangle$ we have

$$\underline{\lim}_{n\to\infty} \frac{a_{n+1}}{a_n} \le \underline{\lim}_{n\to\infty} \sqrt[n]{a_n} \le \overline{\lim}_{n\to\infty} \sqrt[n]{a_n} \le \overline{\lim}_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$\Rightarrow$ if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exist finitely or infinitely then $\lim_{n\to\infty} \sqrt[n]{a_n}$ also exist and $\boxed{\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \sqrt[n]{a_n}}$

(24) D Alembert's Limit:-

If $\langle a_n \rangle$ be a sequence s.t. $a_n \neq 0$. If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$

then (1) If $|l| < 1 \Rightarrow \langle a_n \rangle \longrightarrow 0$

(2) If $|l| > 1 \Rightarrow \langle a_n \rangle$ is divergent.

using D'Almbert test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+3)!}{(n+5)!} \times \frac{(n+4)!}{(2n+1)!}$$

$$= \lim_{n\to\infty} \frac{(2n+3)(2n+2)(2n+1)!}{(n+5)(n+4)!} \times \frac{(n+4)!}{(2n+1)!}$$

$$= \infty > 1$$

$\lim_{n\to\infty} a_n$ = is divergent

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$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} (b_n)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left(\frac{\pi^n}{n^{100}}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\pi}{(n^{\frac{1}{n}})^{100}}$$

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \pi > 1$$

Here $\langle b_n \rangle$ is divergent

$c_n = \dfrac{\log n^{10}}{\sqrt{n}} = \dfrac{10 \log n}{\sqrt{n}}$

c_n is convergent

$\sqrt{n} > \log n$

Here $\dfrac{\log n}{\sqrt{n}} < 1$

$d_n = \dfrac{n^4}{(n+1)!}$

$$\lim_{n\to\infty} \frac{d_{n+1}}{d_n} = \lim_{n\to\infty} \frac{(n+1)^4}{(n+2)!} \times \frac{(n+1)!}{n^4} = \lim_{n\to\infty} \frac{1}{n+2} = 0 < 1$$

$$\lim_{n\to\infty} d_n = 0$$

$\{d_n\}$ is convergent

$\Rightarrow$ $\{c_n\}$ and $\{d_n\}$ is convergent.